臺灣大學數學系

九十學年度第一學期碩博士班資格考試試題

代數

1

Let $M_{n\times n}(\mathbb{C})$ denote the space of complex $n\times n$ matrices. Let $A\in M_{n\times n}(\mathbb{C})$ with A diagonalizable. Consider the linear map $T_A: M_{n\times n}(\mathbb{C})\to M_{n\times n}(\mathbb{C})$ given by

$$T_A(X)=AX-XA,\quad X\in M_{n\times n}(\mathbb{C}).$$

Show that T_A is diagonalizable. What are the eigenvalues of T_A? What are the eigenvectors of T_A?

2

Let A and B be $n\times n$ matrix over $\mathbb{R}$. Suppose that B is invertible and $A^tBA=B$. Prove that λ is an eigenvalue of A if and only if $\frac{1}{\lambda}$ is an eigenvalue of A.

3

Let A be an $n\times n$ complex matrix. Let Tr denote the trace of a matrix. Suppose that $\mathrm{Tr}(A)=\mathrm{Tr}(A^2)=\cdots=\mathrm{Tr}(A^n)=0$. Prove that A is nilpotent.

4

Let G be a finite group and $g\in G$. Let C_g denote the conjugacy class of g and $Z_g=\{x\in G|gx=xg\}$. a. Show that $|C_g|=\frac{|G|}{|Z_g|}$.

b. Suppose $G=S_n$ is the symmetric group in n letters. Describe the conjugacy classes of G.

c. Suppose that $n\geq 3$ and $g=(1,2,3)$. What is $|C_g|$?

5

Let G be a finite group and H a subgroup of index n such that H contains no non-trivial normal subgroup of G. Prove that G may be embedded into S_n.

6

Let R be a commutative Noetherian ring and X an indeterminate. Prove that $R[X]$ is Noetherian.

7

Let R be a commutative ring with 1 and $I_1, I_2, \cdots, I_n$ be ideals of R such that

$I_i + I_j = R$, for $i \neq j$. Prove that the canonical map $R \to \bigoplus_{i=1}^{n} R/I_i$ is a surjective ring homomorphism.

8

Let F be a field and K a field extension of F such that $[K : F] = 2$. Prove that if $\mathrm{char} F \neq 2$, then K is a normal field extension of F.

9

Compute the Galois groups of the following polynomials over $\mathbb{Q}$:

a. $x^3 - 3x + 1$.

b. $x^3 - 3x + 3$.