臺灣大學數學系

八十九學年度第二學期碩博士班資格考試試題

代數

選作6題. $\mathbb{Z}$: integers; $\mathbb{R}$: real numbers; $GL_n(\mathbb{R})$: the group of invertible matrices; $SL_n(\mathbb{R})$:the group of matrices with determinant 1.

1.
Consider a system of linear equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n. \end{aligned}$$

$a_{ij}, b_k \in \mathbb{Z}$. Prove or disprove:

(1)
This system has a rational solution if det $A \neq 0$.

(2)
If the system has a rational solution, then it also has an integer solution.

2.
Let $\phi : F^n \to F^m$ be left multiplication: $X \mapsto AX, A \in M_{m\times n}(F)$. Prove the following are equivalent:

(1)
A has a right inverse B such that $AB = I$.

(2)
ϕ is surjective.

(3)
There is an $m \times m$ minor of A whose determinant is not zero.

3.
Let G be a group containing cyclic normal subgroups of order 10 and 21 respectively. Prove that G contains a cyclic normal subgroup of order 210.

4.
Prove that the matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are conjugate elements in the group $GL_n(\mathbb{R})$, but that they are not conjugate when regarded as elements of $SL_n(\mathbb{R})$.

5.

Determine all ideals of the ring $\mathbb{R}[[t]]$ of formal power series with real coefficients.

6.

Let p be a prime, and $A \in M_n(\mathbb{Z})$ such that $A^p = I$, but $A \neq I$. Prove that $n \geq p-1$.

7.

Let a, b be complex numbers of degree 3 over $\mathbb{Q}$, and let $K = \mathbb{Q}(a, b)$. Determine the possibilities for $[K : \mathbb{Q}]$.

8.

Let G be a finite group. Prove that there exist a field F and a Galois extension K of F whose Galois group is G.