臺灣大學數學系

八十九學年度第一學期碩博士班資格考試試題

代數

Do 6 out of the following 8 problems, including at least one from each set.

I.

1. Show that a group of order 385 contains a normal subgroup of order 77.

2. Let p be a prime and C_p the cyclic group of order p. How many subgroups of order p are there in $G = C_p \oplus C_p \oplus C_p$? Construct another abelian group that is not isomorphic to G but contains the same number of subgroups of order p as G.

II.

1. Let R be the ring of all $n \times n$ matrices over a field F. Show that R is a simple ring, that is, R has no ideal other than 0 or R.

2. Let R be a ring in which thet equation $ax = b$ is always solvable in R for $a, b \in R$ with $a \neq 0$. Show that R is a division ring if R contains more than one elements.

III.

1. Let F be an infinite field and $K = F(x)$ where x is an indeterminate over F. Show that K is Galois over F, that is, the only subfield of K that is fixed by all F-automorphisms of K is F.

2. Find the Galois group of $x^4 + 2$ over the field $\mathbb{Q}$ of rationals.

IV.

1. Let G be the group of all invertible $n \times n$ matrices over a field F. Show that any matrix in G that commutes with all the matrices in G must be a nonzero scalar matrix.

2. Let T be a linear transformation of a finite-dimensional vector space over a field of characteristic 0. Show that $T^m = 0$ for some integer m if T^i has trace 0 for all

$i \geq 1$.