臺灣大學數學系

八十六學年度第二學期碩博士班資格考試試題

微分方程式

25 points each.

1. Solve the following PDEs.
    1. $e^x u_y - u u_x = 0, u(x,0) = -e^x$
    2. $$\begin{aligned} u_{tt} - u_{xx} &= x^2 \text{ in } \mathbb{R} \times [0, \infty) \\ u(x,0) &= u_t(x,0) = 0. \end{aligned}$$
2.
    1. Show that for $n = 3$ the general solution of $u_{tt} - c^2 \Delta u = 0$ with spherical symmetry about the orgin has the form
    $$u = \frac{F(r+ct) + G(r-ct)}{r}, \quad r = |x|$$
    with suitable F, G.
    2. Show that the solution with initial data of the form
    $$u = 0, u_t = g(r) \ (g = \text{even function of} r)$$
    is given by
    $$u = \frac{1}{2cr} \int_{r-ct}^{r+ct} \rho g(\rho)\, d\rho.$$
3. Let Ω be a bounded smooth domain in $\mathbb{R}^n$. Assume $u \in C^2(\bar{\Omega})$ is a solution of
$$\Delta u = 0 \text{ in } \Omega, \ u = f \text{ on } \partial\Omega.$$
    1. Show that
    $$\int_\Omega |\nabla u|^2 \, dx \leq \int_\Omega |\nabla v|^2 \, dx$$
    for any $v \in C^1(\bar{\Omega})$ with boundary values f.
    2. Let $B_{\xi,\rho} = \{x : |x - \xi| < \rho\} \subset \Omega$. Show that
    $$u(\xi) = \frac{1}{\omega_n \rho^{n-1}} \int_{\partial B_{\xi,\rho}} u(x) \, ds_x,$$

where $\omega_n\rho^{n-1}$ is the surface area of $\partial B_{\xi,\rho}$ and ds_x is the surface element on $\partial B_{\xi,\rho}$.

4. Let $K(x,y,t)=(4\pi t)^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4t}}$, f be a continuous bounded function on $\mathbb{R}^n$ and $u(x,t)=\int_{\mathbb{R}^n}K(x,y,t)f(y)\,dy$.
   1. Show that
   $$K(x,0,s+t)=\int_{\mathbb{R}^n}K(x,y,t)K(y,0,s)\,dy$$
   for $s>0, t>0$.
   2. Show that u satisfies $u_t=\Delta u$ for $t>0$ and $\lim_{(z,s)\to(x,0+)}u(z,s)=f(x)$.