# 臺灣大學數學系

## 八十六學年度第一學期碩博士班資格考試試題

## 代數

1. Prove that $\mathbb{Q}$ has no proper subgroups of finite index. Deduce that $\mathbb{Q}/\mathbb{Z}$ has no proper subgroups of finite index, where $\mathbb{Q}$ is the additive group of rational numbers and $Z$ is the additive group of integers.
2. Let $G$ be a finite group of odd order and $x$ a nonidentity element in $G$. Prove that $x$ and $x^{-1}$ are not conjugate in $G$.
3. Let $R = \mathrm{M}_n(F)$, the $n$ by $n$ matrix ring over a field $F$, $n > 1$ and $a, b$ nonzero elements in $R$. Show the following statements:
    1. $\mathrm{rank}(a) = \mathrm{rank}(b)$ if and only if $\dim_F aR = \dim_F bR$.
    2. $axa = a$ for some $x \in R$.
    3. $aR = eR$ for some element $e = e^2 \in R$.
4. Prove that $\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$ is a Euclidean domain with the norm $N(a + b\sqrt{-2}) = a^2 + 2b^2$, where $\mathbb{Z}$ is the ring of integers. Also, find $q, r \in \mathbb{Z}[\sqrt{-2}]$ such that $35 + 23\sqrt{-2} = (5 + 7\sqrt{-2})q + r$ with $N(r) < 123$.
5. Let $K$ be the splitting field of $X^4 - 7$ over $\mathbb{Q}$, the field of rational numbers. Find the Galois group $G$ of $K$ over $\mathbb{Q}$ and describe the correspondence between the subgroups of $G$ and the subfields of $K$.