Syllabuses for Honor Courses

Algebra I & II

Algebra is a fundamental part of the language of mathematics. Algebraic methods are used in all areas of mathematics. We will fully develop all the key concepts. We will also examine in more detail all the basic algebraic structures, various constructions in algebra. Level of abstraction increases as the full year course goes on. This course sets up a bridge between classical high-school algebra problems to the more advanced frontiers of modern mathematical researches. Additional topics in commutative algebra, or linear representations of finite groups will be covered if time is permitted.

Part 1. Group Theory.

Permutation groups, Cyclic groups, Abelian groups, Free groups, Group homomorphisms, Quotient groups, Group actions, Sylow theorems, Fundamental theorem of finite abelian groups, Direct product and semidirect product, Group extensions, Finite simple groups, Solvable groups.

Part 2. Ring Theory.

Rings, Ideals, Matrix rings, Integral domains, Rings of fractions, Euclidean domains, Principal ideal domains, Unique factorization domains, Rings of quadratic algebraic integers, Polynomial rings, Power series rings, Resultant, Hilbert basis theorem, Gröbner basis.

Part 3. Field Theory.

Field extensions, Fundamental theorem of Galois theory, Finite fields, Solvability by radicals, Cyclotomic extension, Kummer extension, Hilbert's theorem 90, Galois resolvent, Traces, Norms.

Part 4. Module Theory.

Tensor product, Free modules, Projective modules, Injective and flat modules, Symmetric and exterior algebras, Modules over PID, Introduction to Homological algebra.

Part 5. Additional Topics* (with emphasizes on examples).

*Introduction to Commutative Algebra:
Noetherian rings and Artinian rings, Associated primes and Primary decomposition, Going-up and Going-down theorems, Dedekind domains.

*Introduction to the representation theory of finite groups:
Linear representations, Characters, Induced characters, Induced representations.

Selected References:

M. Artin : Algebra.
Dummit-Foote: Abstract Algebra.
N. Jacobson: Basic Algebra I.
S. Lang: Undergraduate Algebra.
Van der Waerden: Algebra I, 7th edition.

Analysis I & II

(The First Semester)
Chapter 1. $\mathbb{R}^n$ and its topology (7 weeks)

§1.1 Introduction

Metric on $\mathbb{R}^n$, Schwartz inequality, limits, Cauchy sequence, series, completeness of $\mathbb{R}^n$, countable and uncountable sets. Examples.

§1.2 Topology of $\mathbb{R}^n$

1. Open set, interior of a set, closed set, closure, accumulation points, boundary. Examples.
2. Continuous functions (Examples, including monotone), uniform convergence and power series.

§1.3 Compact and connected sets. (1 week)

1. Compact sets, The Heine-Borel (Bolzano-Weierstrass) theorem and open covering.
2. Connected set, one-dimensional classification, path-connectedness, applications to continuous functions (Intermediate theorem). Examples. (§§1.1-1.3 for 3 weeks)

§1.4 Metric space

1. $\mathbb{R}^n$ (in ℓ^p)
2. The space of continuous functions, Ascoli-Arzelà theorem, Stone-Weierstrass theorems (compactness revisited).
3. Fixed point theorem.
   (a) Contraction maps.
   (b) Applications to O.D.E: existence and uniqueness, continuity on initial data, local stability in 2×2 system. (2 weeks)
   (c) Brouwer fixed points (optional).
4. Baire category theorem and its applications.

Chapter 2. Differentials (6 weeks)

§2.1

1. Overview (one-dimension) Rolle's theorem and mean-value theorem. Applications.
2. Linear transformation, Differential (definition) and example (with emphasis on geometry)
3. Partial derivatives and continuous P.D $\rightarrow$ differentials
4. The chain rule. The determinant of its differential (for $\mathbb{R}^n \rightarrow \mathbb{R}^n$)

§2.2 Higher order differentials. Partial derivatives. Taylor expansions.

§2.3 Implicit function theorem and its variations. Examples.

§2.4 Local extrema and Lagrange multipliers. Applications. Examples.

§2.5 Sard's theorem.

Chapter 3. Riemann Integral of multi-variables (1.5 weeks)

1. Definition, change of variable formulas (for continuous function). Examples.
2. Surface integral and line integral. Stokes' theorem. (Applications)
3. Lebesgue's criterion for the existence of Riemann integral. (1 week, the next semester)

(The Second Semester)

Chapter 4. Lebesgue integrals (8 weeks)

§4.1 Measure and measurable functions. (1 week)

§4.2 Lebesgue integrals. Discuss the difference between Lebesgue integrable and Riemann integrable functions. L^1 space.

§4.3 Convergence theorem:

1. Monotone, dominated convergence, and the Fatou lemma.
2. (Convergence almost everywhere)
3. Counterexamples (Weak convergence does not imply the convergence of the integral).
4. Convergence implies almost uniform convergence. (2 weeks for §§4.2, 4.3)

§4.4 The Fubini theorem and its applications. (1 week)

§4.5 Differentiation and integration. (4 weeks)

1. Hardy-littlewood maximal function, Vitali covering.
2. Lebesgue's point.
3. Differentiation under the integral.
4. Convolution.
5. Approximation to the identity.
6. BV functions.
7. Absolutely continuous function, etc.

Chapter 5. Fourier theory (Introduction) (5 weeks)

1. $L^2[0,1]$, Fourier series, Parseval's theorem, and the completeness, examples.
2. $L^2(\mathbb{R})$, Fourier transformation and the inverse Fourier transformation, the Parseval theorem and the Poisson summation, examples.
3. Applications to Laplace equation, Heat equation, and hyperbolic equation.

Complex Analysis

Complex function theory is a valuable tool used in many branches of pure, applied mathematics and natural sciences, including geometry, number theory, ordinary differential equations, partial differential equations and various topics in physics and engineering. A basic course shall enable student to understand the concept of complex analyticity, to use residue calculus for evaluation of integrals, to understand conformal mappings, and to learn some additional topics selected from Riemann mapping theorem, special/elliptic functions, prime number theorem, complex dynamical systems, etc.

Contents: (those with * are optional)

Chapter 1: Cauchy theory (4 weeks)

Cauchy-Goursat theorem and Cauchy's integral formula, singularities of analytic functions and Laurent series, residue theorem and its applications: maximum modulus principle, open mapping theorem, argument principle, Rouche's theorem and the evaluation of integrals.

Chapter 2: Entire functions (4 weeks)

Jensens's formula, Weierstrass infinite products, Hadamard's factorization theorem, analytic continuation of gamma and zeta functions, prime number theorem.

Chapter 3: Conformal mappings (4 weeks)

Conformal mapping on basic domains and Schwarz Lemma, hyperbolic geometry Riemann mapping theorem, Schwartz reflections and boundary behavior Schwartz-Christoffel integrals, harmonic functions and the Dirichlet problem for multiply-connected regions.

Chapter 4: Special topics-select two from the following: (4 weeks)

*Compact family of meromorphic functions, theorem of Montel and Picard, applications

*Introduction to complex dynamical systems, Julia sets,

*Introduction to elliptic functions and modular functions,

*Introduction to linear ODE, monodromy groups, and analytic continuations.

Selected references:

1. Stein, E.M., Shakarchi, R., "Complex analysis"
2. Lang, S., "Complex analysis", GTM 103
3. Ahlfors, L., "Complex analysis"
4. Whittaker-Watson, "Modern Analysis"
5. Gamelin, T.W., "Complex Analysis"

Geometry

Modern geometry is widely used in pure and applied mathematics as well as in physics and engineering. This course starts with the classical theory of curves and surfaces in three dimensional Euclidean spaces, which provides the most visualizable and fundamental examples for applications as well as for higher dimensional theories. It then moves to the foundation of differentiable manifolds and vector bundles. The aim of this course is to provide sufficient background for future study on Riemannian geometry, symplectic geometry, differential topology, as well as global analysis (PDE, ODE) on manifolds.

Syllabus (those with * are optional)

1. Theory of surfaces (6 weeks)
1.1 Plane curves and space curves, Frenet frame
1.2 First and second fundamental forms of surfaces in Euclidean spaces
1.3 Abstract surfaces, Levi-Civita connection and parallel translations
1.4 Gaussian curvature, holomomy angle, Gauss-Bonnet theorem and its applications
1.5 Gauss-Codazzi equations, mean curvature, minimal surfaces, and *the Weierstrass representations

2. Differentiable manifolds (5 weeks)
2.1 Definitions and examples of differentiable manifolds and differentiable maps
2.2 Tangent bundles, tangent maps, and Sard's theorem
2.3 Embeddings, immersions, submersions, and Whitney's theorem
2.4 Flow of a vector field, Lie derivatives, and Frobenius' theorem
2.5 Introduction to Lie groups and Lie algebras through examples (classical groups)

3. Differential forms and vector bundles (5 weeks)
3.1 Differential forms and Stokes' theorem
3.2 De Rham cohomology, Poincare duality, and de Rham theorem
3.3 Connections and curvature on vector bundles
3.4* Chern-Weil theory of characteristic classes
3.5* Flat bundles and representations of the fundamental group

Bibliography

D. Struik: Lectures on classical differential geometry
M. Do Carmo: Differential geometry of curves and surfaces
M. Hirsch: Differential topology
M. Spivak: A comprehensive introduction to differential geometry vol I and II
J. Milnor: Characteristic classes
R. Bott and L. Tu: Differential forms in algebraic topology