有關 $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots=\frac{\pi^2}{6}$

早在十八世紀時, 數學家Euler就知道無窮級數$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, 關於他那時候的想法, 我們可在以下網站得到一些資訊:

`http://episte.math.ntu.edu.tw/articles/mm/mm_02_3_09/index.html`

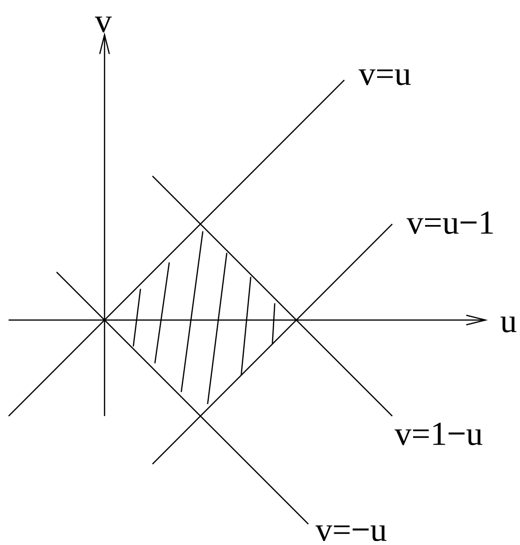
現在, 我們利用微積分的方法重新得到此級數和. 用到的觀念只是變數變換: 考慮在$0\le x\le 1$, $0\le y\le 1$上積分

$$\iint\frac{1}{1-xy}dxdy=\iint(1+xy+x^2y^2+\cdots)dxdy=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

另外作變數變換$x=u+v$, $y=u-v$, 得到 $|J|=\left|\begin{array}{cc}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{array}\right|=|-2|=2$
所以

$$\iint\frac{1}{1-xy}dxdy=\iint\frac{2dudv}{1-u^2+v^2}$$

積分範圍如下:



$$\begin{aligned} & \iint\frac{2dudv}{1-u^2+v^2} \\ = & \int_0^{u=\frac{1}{2}}\int_{v=-u}^{v=u}\frac{2dv}{1-u^2+v^2}du \quad +\int_{u=\frac{1}{2}}^{u=1}\int_{v=u-1}^{v=1-u}\frac{2dv}{1-u^2+v^2}du \\ = & A \quad +B\end{aligned}$$

$$
\begin{aligned}
A &= 2\int_{u=0}^{u=\frac{1}{2}} \left[\frac{1}{\sqrt{1-u^2}}\tan^{-1}(\frac{v}{\sqrt{1-u^2}})\right]\Big|_{v=-u}^{v=u} du \\
&= 4\int_{u=0}^{u=\frac{1}{2}} \frac{1}{\sqrt{1-u^2}}\tan^{-1}(\frac{u}{\sqrt{1-u^2}}) du
\end{aligned}
$$

令 $\theta = \tan^{-1}(\frac{u}{\sqrt{1-u^2}}) \Rightarrow \tan\theta = \frac{u}{\sqrt{1-u^2}}$

$\sec^2\theta d\theta = d(\frac{u}{\sqrt{1-u^2}}) = -\frac{1}{2}\frac{-2u\cdot u\cdot du}{(1-u^2)^{\frac{3}{2}}} + \frac{du}{(1-u^2)^{\frac{1}{2}}} = \frac{1}{(1-u^2)^{\frac{3}{2}}}du$

又因 $\sec^2\theta = 1+\tan^2\theta = 1+\frac{u^2}{1-u^2} = \frac{1-u^2+u^2}{1-u^2} = \frac{1}{1-u^2}$

$\therefore d\theta = \frac{1}{(1-u^2)^{\frac{1}{2}}}du$

於是$A = 4\cdot\int_{\theta=0}^{\theta=\frac{\pi}{6}} \theta d\theta = 4\cdot\frac{1}{2}\cdot(\frac{\pi}{6})^2 = 2\cdot(\frac{\pi}{6})^2$

$$
\begin{aligned}
B &= 2\cdot\int_{u=\frac{1}{2}}^{u=1}\int_{v=u-1}^{v=1-u} \frac{2dv}{1-u^2+v^2} du \\
&= 2\cdot\int_{u=\frac{1}{2}}^{u=1} \left[\frac{1}{\sqrt{1-u^2}}\tan^{-1}(\frac{v}{\sqrt{1-u^2}})\right]\Big|_{v=u-1}^{v=1-u} du \\
&= 4\cdot\int_{u=\frac{1}{2}}^{u=1} \frac{1}{\sqrt{1-u^2}}\tan^{-1}(\frac{1-u}{\sqrt{1-u^2}}) du
\end{aligned}
$$

令 $\theta = \tan^{-1}(\frac{1-u}{\sqrt{1-u^2}}) \Rightarrow \tan\theta = \frac{1-u}{\sqrt{1-u^2}}$

$\sec^2\theta d\theta = d(\frac{1}{\sqrt{1-u^2}}) - d(\frac{u}{\sqrt{1-u^2}}) = \frac{udu}{(1-u^2)^{\frac{3}{2}}} - \frac{du}{(1-u^2)^{\frac{3}{2}}} = \frac{(u-1)du}{(1-u^2)^{\frac{3}{2}}}$

又因 $\sec^2\theta = 1+\tan^2\theta = 1+\frac{(1-u)^2}{1-u^2} = \frac{2-2u}{1-u^2}$

$\therefore d\theta = -\frac{1}{2\sqrt{1-u^2}}du$

於是$B = -8\int_{\theta=\frac{\pi}{6}}^{\theta=0} \theta d\theta = 4\cdot(\frac{\pi}{6})^2$

$$
\underline{\therefore \iint \frac{1}{1-xy}dxdy = A+B = \frac{\pi^2}{6} = \sum_{k=1}^{\infty}\frac{1}{k^2}}
$$