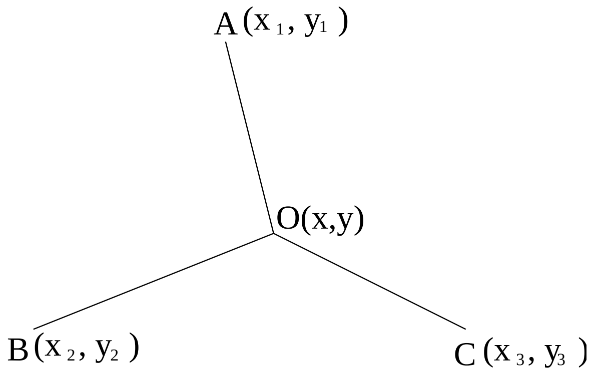
到三角形三頂點之和何時最小

平面上A, B, C三點不共線, 找一點O, 使得$\overline{OA}+\overline{OB}+\overline{OC}$ 為最小?



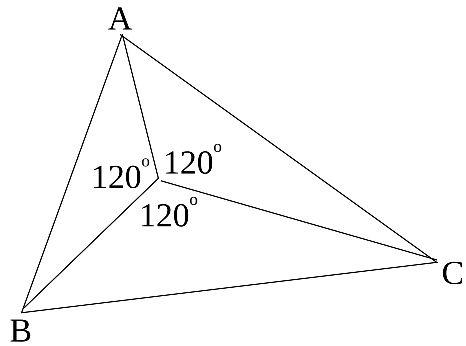
解: 給定坐標: $A:(x_1,y_1)$, $B:(x_2,y_2)$, $C:(x_3,y_3)$, $O:(x,y)$, 其中 x,y為變數, 則:$\overline{OA}+\overline{OB}+\overline{OC}=f(x,y)=\sqrt{(x-x_1)^2+(y-y_1)^2}$ $+\sqrt{(x-x_2)^2+(y-y_2)^2}$ $+\sqrt{(x-x_3)^2+(y-y_3)^2}$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x}=\frac{x-x_1}{\sqrt{(x-x_1)^2+(y-y_1)^2}}+\frac{x-x_2}{\sqrt{(x-x_2)^2+(y-y_2)^2}}+\frac{x-x_3}{\sqrt{(x-x_3)^2+(y-y_3)^2}} \\ \frac{\partial f}{\partial y}=\frac{y-y_1}{\sqrt{(x-x_1)^2+(y-y_1)^2}}+\frac{y-y_2}{\sqrt{(x-x_2)^2+(y-y_2)^2}}+\frac{y-y_3}{\sqrt{(x-x_3)^2+(y-y_3)^2}} \end{cases}$$

$f(x,y)$ 最小值可能發生之處為 $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0$的點或偏導數不存在的$A,B,C$三點. 若$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0$ 可解 $\Leftrightarrow$ 向量方程

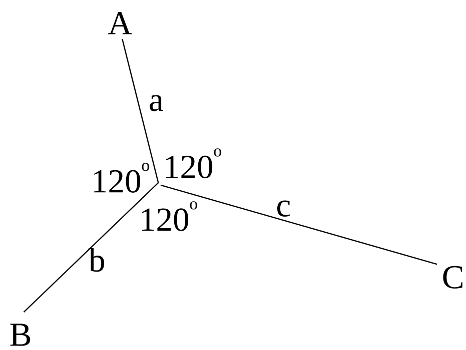
$$\underbrace{\begin{pmatrix} \frac{x-x_1}{\sqrt{(x-x_1)^2+(y-y_1)^2}} \\ \frac{y-y_1}{\sqrt{(x-x_1)^2+(y-y_1)^2}} \end{pmatrix}}_{\downarrow \atop \overrightarrow{OA}\text{的單位向量}}+\underbrace{\begin{pmatrix} \frac{x-x_2}{\sqrt{(x-x_2)^2+(y-y_2)^2}} \\ \frac{y-y_2}{\sqrt{(x-x_2)^2+(y-y_2)^2}} \end{pmatrix}}_{\downarrow \atop \overrightarrow{OB}\text{ 的單位向量}}+\underbrace{\begin{pmatrix} \frac{x-x_3}{\sqrt{(x-x_3)^2+(y-y_3)^2}} \\ \frac{y-y_3}{\sqrt{(x-x_3)^2+(y-y_3)^2}} \end{pmatrix}}_{\downarrow \atop \overrightarrow{OC}\text{ 的單位向量}}=0$$

可解 $\Leftrightarrow$存在一點O使得$\overrightarrow{OA}$的單位向量$+$ $\overrightarrow{OB}$的單位向量 $+$ $\overrightarrow{OC}$的單位向量 $=0$向量. 三個單位向量和為0, 則三個向量必互相夾120°.

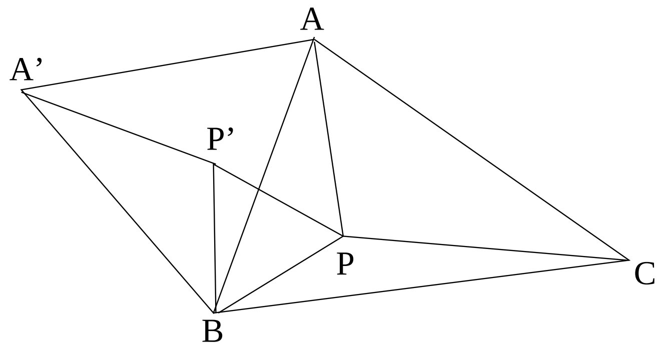


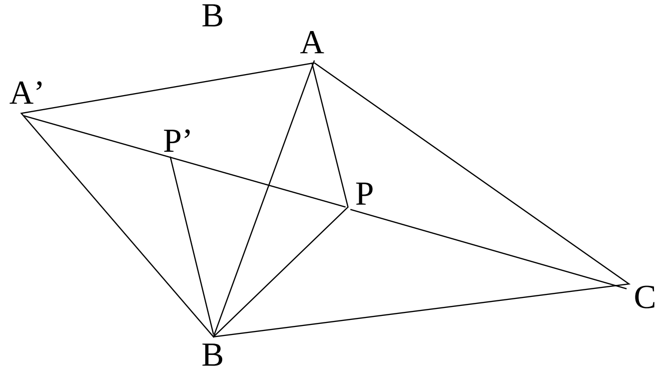
$\Rightarrow\ \angle A, \angle B, \angle C$皆小於$120°$; 反之, 若$\angle A, \angle B, \angle C$皆小於$120°$, 必可在A,B,C內部找到O點.

$\vec{OA}, \vec{OB}, \vec{OC}$夾角皆為$120°$ $\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$可解$\Leftrightarrow \angle A, \angle B, \angle C$皆小於$120°$.



令$|\vec{OA}| = a, |\vec{OB}| = b, |\vec{OC}| = c$, 若$\angle A, \angle B, \angle C$皆小於$120°$, 則$f(x,y)$最小值發生在 A,B,C或 $\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ 的解O上. 而$f(x,y)$在A的值為$\overline{AB} + \overline{AC} = \sqrt{a^2+b^2+ab} + \sqrt{a^2+c^2+ac} \geq \sqrt{b^2+ab+\frac{1}{4}a^2} + \sqrt{c^2+ac+\frac{1}{4}a^2} = (b+\frac{1}{2}a) + (c+\frac{1}{2}a) = a+b+c = f(x,y)$在O的值. 同理, $f(x,y)$在B, C的值皆大於$f(x,y)$在O的值. $f(x,y)$的最小值存在, 可能發生的點僅A, B, C, O, 又在四點中, O的值最小, $\therefore f(x,y)$最小值發生在O. O滿足$\vec{OA}, \vec{OB}, \vec{OC}$夾角$120°$ 若$\angle A, \angle B, \angle C$其中一角大於或等於$120°$則$f(x,y)$最小值可能發生在$A, B, C$值分別為$\overline{AB}+\overline{AC}$, $\overline{BA}+\overline{BC}$, $\overline{CA}+\overline{CB}$, 即$s-\overline{BC}, s-\overline{AC}, s-\overline{AB}$, $s = \overline{AB} + \overline{BC} + \overline{AC}$, 又角A為最大角, $\overline{BC}$為最大邊, $\therefore s - \overline{BC}$ 最小. $\therefore f(x,y)$最小值發生在A.

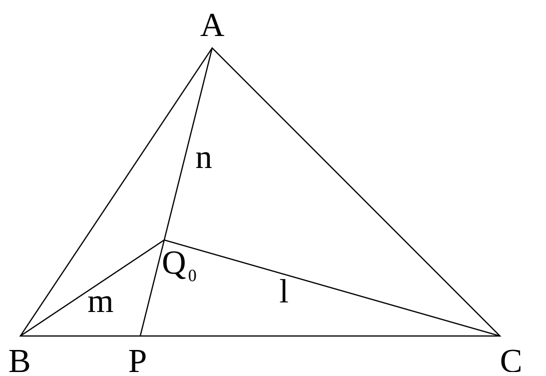




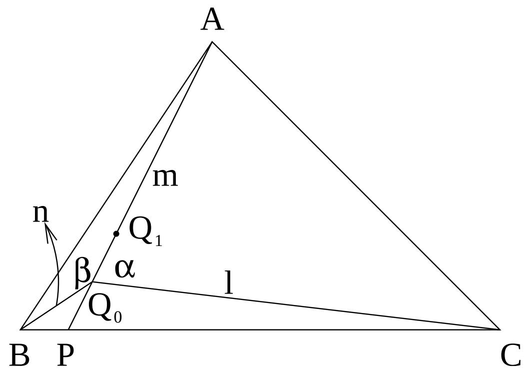
註: 當$\angle A, \angle B, \angle C < 120°$本題有一純幾何的做法作正$\triangle AA'B$, 正$\triangle PP'B$, 則 $\triangle APB \cong \triangle A'P'B'$, $\overline{AP} + \overline{BC} + \overline{CP} = \overline{A'P'} + \overline{PP'} + \overline{PC}$若要最小, 則須選取P使得P, P’ A’, C在同一直線上. 此時$\angle P'PB = \angle A'AB$, A’APB為圓的內接四邊形. $\therefore \angle APA' = \angle ABA' = 60°$
$\therefore \angle APB = \angle BPC = \angle CPA = 120°$ 此一P點一般稱為費馬點

解二(不用偏微分)

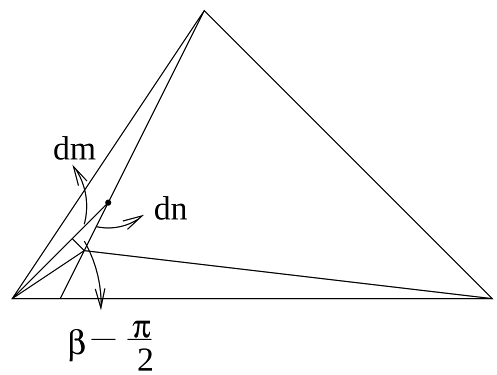
把問題改變一下: 如圖:



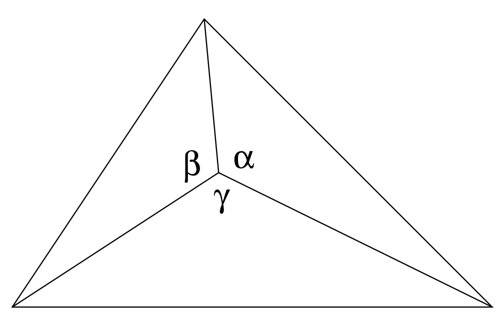
在AP上找一點Q, 使$\overline{AQ} + \overline{BQ} + \overline{CQ}$最小, 先任取一點 Q_0,



讓Q_0變動一點點到Q_1來看看, 則到三點距離和的變化是$dn + dl + dm$. 不妨設β是鈍角, α是銳角, 則有$dm > 0$, $dl < 0$, $dn < 0$, 且 $(dn)\cos\alpha = dl$, $(dn)\cos\beta = dm$, $dn < 0$, $\cos\alpha > 0$, $\cos\beta < 0$,



因此要求: $dn{+}dl{+}dm = 0$ 就得到 $1{+}\cos\alpha{+}\cos\beta = 0$, 或$\cos\alpha{+}\cos\beta = -1$. 如果不要求$Q$, 一定要在特定的直線AP上, 但是仍然要求$\overline{AQ}+\overline{BQ}+\overline{CQ}$最小,



如此一來, 當然有$\cos\alpha + \cos\beta = \cos\alpha + \cos\gamma = \cos\beta + \cos\gamma = -1$, 因此 $\alpha = \beta = \gamma = 120°$.