Stirling 近似公式: $n! \sim \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n+\frac{1}{12n}}$

給定兩數列$\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, 若 $\lim_{n\to\infty}\frac{a_n}{b_n}=1$, 則我們記為 $a_n \sim b_n$

$$\begin{aligned}\ln n! &= \sum_{k=1}^{n}\ln k = \sum_{k=1}^{n}\int_1^k \frac{1}{x}dx = \sum_{k=1}^{n}(n-k)\int_k^{k+1}\frac{1}{x}dx \\ &= \int_1^n \frac{n}{x}dx - \int_1^n \frac{[x]}{x}dx = (n+\tfrac{1}{2})\ln n - n + 1 + \int_1^n \frac{x-[x]-\frac{1}{2}}{x}dx\end{aligned}$$

因為 $\int_1^x t-[t]-\frac{1}{2}dt$有界, 且$\frac{1}{x}$遞減到零, 利用部分積分我們知道$\int_1^\infty \frac{x-[x]-\frac{1}{2}}{x}dx$收斂. 令$C$為$\exp(1+\int_1^\infty \frac{x-[x]-\frac{1}{2}}{x}dx)$, 則 $n! \sim Cn^{n+\frac{1}{2}}e^{-n}$.

由 Wallis's Formula : $\lim_{n\to\infty}\left(\frac{2\times 2}{1\times 3}\times\cdots\times\frac{2n\times 2n}{(2n-1)\times(2n+1)}\right)=\frac{\pi}{2}$ (Courant and John 微積分一冊282頁), 我們可得

$$\begin{aligned}\frac{\pi}{2} &\sim \frac{2\times 2}{1\times 3}\times\cdots\times\frac{2n\times 2n}{(2n-1)\times(2n+1)} = \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!} \\ &\sim \frac{2^{4n}(Cn^{n+\frac{1}{2}}e^{-n})^4}{C(2n)^{2n+\frac{1}{2}}e^{-2n}\times C(2n+1)^{2n+1+\frac{1}{2}}e^{-2n-1}} \\ &\sim \frac{C^2}{4}\end{aligned}$$

(註:上式中的$\sim$是一個等價關係)

所以$C=\sqrt{2\pi}$, 因此我們得到 Stirling 近似公式: $n! \sim \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}$. 令 $\varepsilon_n = -\int_n^\infty \frac{x-[x]-\frac{1}{2}}{x}dx$, 我們可得 $n! \sim \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n+\varepsilon_n}$.

$$\begin{aligned}\varepsilon_n &= -\sum_{k=n}^{\infty}\int_k^{k+1}\frac{x-[x]-\frac{1}{2}}{x}dx = \sum_{k=n}^{\infty}(-1+(k+\tfrac{1}{2})\ln\tfrac{k+1}{k}) \\ &= \sum_{k=n}^{\infty}(-1+\tfrac{1}{2}(2k+1)\ln\frac{1+\frac{1}{2k+1}}{1-\frac{1}{2k+1}}) = \sum_{k=n}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{(2\ell+1)(2k+1)^{2\ell}}\end{aligned}$$

因此我們得到

$$\frac{1}{12(n+\frac{1}{2})} < \sum_{k=n}^{\infty}\frac{1}{3}\frac{1}{(2k+1)^2} < \varepsilon_n < \sum_{k=n}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{3}\frac{1}{(2k+1)^{2\ell}} = \frac{1}{12n}$$

我們可得$n!$的上下界.

$$\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n+\frac{1}{12(n+\frac{1}{2})}} < n! < \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n+\frac{1}{12n}} < \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}(1+\frac{1}{12n-1})$$

若令 $f(n)=\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n+\frac{1}{12n}}$, 則$f(10)=3628810.05$, 與$10!=3628800$的誤差率小於 0.0003%.

也請參考Courant and John一冊504頁.