Problem: $a\ ,\ n \in N,\ a \neq 10^i, i \in Z, i \geq 0,\ a^n = \sum_{k=0}^{m(n)} a_k \cdot 10^k$ and set $D_i^N = \{n | a_{m(n)} = i,\ 0 \leq n \leq N\}$, $1 \leq i \leq 9$. We want to prove that

$$\lim_{N \to \infty} \frac{|D_i^N|}{N} = \log(i+1) - \log(i)$$

Definition : $r \notin Q, m, n \in N, \{nr\} = nr - [nr]$

Lemma : $\{nr\} = \{mr\} \Rightarrow n = m.$

Proof : $\{nr\} = \{mr\} \Rightarrow (m-n)r \in Z$, if $m \neq n$, $\Rightarrow r \in Q$. $(\rightarrow\leftarrow)$

Theorem : $\{nr\}$ has uniform distribution in (0,1).

見Hardy and Wright, An Introduction to the Theory of Numbers. P390, 定理445.