BASE CHANGE AND TRIPLE PRODUCT L-SERIES

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Contents

1. Introduction

Let F be a number field with adèle ring $\mathbb{A}$, K a quadratic extension of F with adèle ring $\mathbb{K}$ and D a quaternion algebra over F. Let $D_K = D \otimes_F K$ be a quaternion algebra over K. Let ϵ_K be the character of $F^\times \backslash \mathbb{A}^\times$ of order 2 corresponding to K via class field theory. Let π_i be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ with central character ω_i. We denote the base change lift of π_i to $\mathrm{GL}_2(\mathbb{K})$ by $\pi_{i,K}$. We impose on π_i and D the following conditions:

(Cent) $\omega_1\omega_2\omega_3 = \epsilon_K$;

(Cusp) $\pi_{i,K}$ is cuspidal;

(JL) there exists an irreducible automorphic representation $\pi_{i,K}^D$ of $D_K^\times(\mathbb{A})$ associated to $\pi_{i,K}$ by the Jacquet-Langlands correspondence;

(Per) the period integral

$$B_i(\phi_i) = \int_{\mathbb{A}^\times D^\times(F)\backslash D^\times(\mathbb{A})} \phi_i(h)(\omega_i^{-1}\epsilon_K)(\mathrm{N}_{D/F}(h))\,\mathrm{d}h$$

does not vanish for some $\phi_i \in \pi_{i,K}^D$, where $\mathrm{N}_{D/F}$ denotes the reduced norm on D and $\mathrm{d}h = \prod_v \mathrm{d}h_v$ is the Tamagawa measure on $\mathbb{A}^\times \backslash D^\times(\mathbb{A})$.

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(Arc) K is split at all the archimedean places of F.

The assumption (JL) is automatic if $D_K \simeq \mathrm{M}_2(K)$. The assumption (Per) is automatic if $D \simeq \mathrm{M}_2(F)$ (see Proposition 2.13(1)). The assumption (Arc) is made for convenience to simplify the local calculations in §4.

Put $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ and $\Pi_K^D = \pi_{1,K}^D \otimes \pi_{2,K}^D \otimes \pi_{3,K}^D$. One of the purposes of this article is to establish an explicit formula relating the period integral

$$I(\phi_1 \otimes \phi_2 \otimes \phi_3) = \int_{\mathbb{K}^\times D_K^\times(F)\backslash D_K^\times(\mathbb{A})} \phi_1(\Xi)\phi_2(\Xi)\phi_3(\Xi)\,\mathrm{d}\Xi$$

on Π_K^D to the central value $L\big(\frac{1}{2}, \Pi\big) = L\big(\frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3\big)$ of the triple product L-series associated to the Langlands parameters of π_i and the eight-dimensional representation of the L-group of $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$. Here $\mathrm{d}\Xi = \prod_v \mathrm{d}\Xi_v$ is the Tamagawa measure on $\mathbb{K}^\times\backslash D_K^\times(\mathbb{A})$.

For each place v of F we let F_v be the completion of F at v and put

$$D_v = D \otimes F_v, \qquad\qquad D_{K_v} = D \otimes K_v.$$

We will explicitly factorize B_i into local functionals in §2.5. Choose a local invariant form $B_{i,v} \in \mathrm{Hom}_{D_v^\times}(\pi_{i,K_v}^{D_v}, \omega_{i,v}\epsilon_{K_v})$ so that

$$B_i(\phi_i) = \prod_v B_{i,v}(\phi_{i,v})$$

for $\phi_i = \otimes\phi_{i,v} \in \pi_{i,K}^D$, and $B_{i,v}(\phi_{i,v}) = 1$ for almost all v. Put

$$B_v = B_{1,v} \otimes B_{2,v} \otimes B_{3,v}, \qquad L(s, \mathrm{Ad}(\Pi) \otimes \epsilon_K) = \prod_{i=1}^3 L(s, \mathrm{Ad}(\pi_i) \otimes \epsilon_K).$$

We define an element of the space

$$\mathrm{Hom}_{D_{K_v}^\times}(\Pi_{K_v}^{D_v}, \mathbb{C}) \tag{1.1}$$

by the convergent integral

$$\mathbf{I}_v(\phi_v) = \int_{K_v^\times D_v^\times\backslash D_{K_v}^\times} B_v(\Pi_{K_v}^{D_v}(\xi_v)\phi_v)\,\mathrm{d}\xi_v$$

for $\phi_v \in \Pi_{K_v}^{D_v} = \pi_{1,K_v}^{D_v} \otimes \pi_{2,K_v}^{D_v} \otimes \pi_{3,K_v}^{D_v}$, where $\mathrm{d}\xi_v$ is the measure defined by the quotient of $\mathrm{d}\Xi_v$ by $\mathrm{d}h_v$. Let $\epsilon(D_v)$ be either 1 or -1 according as D is split at v or not. Fix a non-trivial additive character $\psi = \prod_v \psi_v$ of $F\backslash\mathbb{A}$. We write

$$\gamma(\Pi_v) = \gamma\left(\frac{1}{2}, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v}, \psi_v\right)$$

for the central value of the triple product gamma factor. Note that $\gamma(\Pi_v)$ is independent of the choice of ψ_v by Remark 1.4(2).

Theorem 1.1. *Assume that $K_v \simeq F_v \times F_v$ if v is archimedean. The functional $\mathbf{I}_v$ is non-vanishing if and only if*

$$\gamma(\Pi_v) \neq -\epsilon_K(-1)\epsilon(D_v).$$

If $K_v \simeq F_v \times F_v$, then $\gamma(\Pi_v) = \varepsilon\big(\frac{1}{2}, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v}, \psi_v\big)$ is the local root number, and Theorem 1.1 is known as the epsilon dichotomy, proved by Prasad in [Pra90]. On the other hand, if K_v is not split at v, then $D_{K_v} \simeq \mathrm{M}_2(K_v)$, and (1.1) is one-dimensional (see Remark 3.4(1)). Two functionals thus constructed satisfy the relation stated in Proposition 4.7. Note that $\gamma(\Pi_v)$ may not be a sign.

Theorem 1.2. *Assume the following conditions:*

- *v is non-archimedean,*
- *ϵ_{K_v} is unramified,*
- *$\epsilon(D_v) = 1$,*
- *$B_v(\phi_v) = 1$ and $\Pi_{K_v}^{D_v}(k)\phi_v = \phi_v$ for $k \in \mathrm{GL}_2(\mathfrak{o}_{K_v} \times \mathfrak{o}_{K_v} \times \mathfrak{o}_{K_v})$,*
- *$\mathrm{d}\xi_v$ is the right invariant measure on $K_v^\times \mathrm{GL}_2(F_v)\backslash \mathrm{GL}_2(K_v)$ which gives $\mathfrak{o}_{K_v}^\times \mathrm{GL}_2(\mathfrak{o}_{F_v})\backslash \mathrm{GL}_2(\mathfrak{o}_{K_v})$ volume 1.*

Then

$$\mathbf{I}_v(\phi_v) = \frac{\zeta_{F_v}(2)^2 L\big(\frac{1}{2}, \Pi_v\big)}{L(1, \mathrm{Ad}(\Pi_v) \otimes \epsilon_{K_v})} \cdot \frac{1 + \gamma(\Pi_v)}{2}.$$

When $\gamma(\Pi_v) \neq -\epsilon_K(-1)\epsilon(D_v)$, we normalize the functional $\mathbf{I}_v$ by setting

$$I_v = \frac{L(1, \mathrm{Ad}(\Pi_v) \otimes \epsilon_{K_v})}{\zeta_{F_v}(2)^2 L\big(\frac{1}{2}, \Pi_v\big)} \cdot \frac{2}{\epsilon(D_v) + \epsilon_K(-1)\gamma(\Pi_v)} \cdot \mathbf{I}_v.$$

Theorem 1.3. *Assume the conditions (Arc), (Cent) and (Cusp). If there exists a quaternion algebra D over F which satisfies $\epsilon(D_v) \neq -\epsilon_K(-1)\gamma(\Pi_v)$ for all v, then such D satisfies the conditions (JL) and (Per), and we have*

$$I = 2^{-3} \cdot \frac{\zeta_F(2)^2 L\big(\frac{1}{2}, \Pi\big)}{L(1, \mathrm{Ad}(\Pi) \otimes \epsilon_K)} \cdot \prod_v I_v$$

as elements of $\mathrm{Hom}_{D_K^\times(\mathbb{A})}(\Pi_K^D, \mathbb{C})$.

Remark 1.4. (1) We define the L-series $L(s, \mathrm{Ad}(\pi_i) \otimes \epsilon_K)$ as the ratio

$$L(s, \mathrm{Ad}(\pi_i) \otimes \epsilon_K) = L(s, \pi_i \times \pi_i^\vee \otimes \epsilon_K)/L(s, \epsilon_K).$$

Since $\pi_i \not\simeq \pi_i \otimes \epsilon_K$ by (Cusp), the L-series $L(s, \mathrm{Ad}(\pi_i) \otimes \epsilon_K)$ has neither zero nor pole at $s = 1$.

(2) Fix $a \in F_v^\times$ and define the character ψ_v^a of F_v by $\psi_v^a(x) = \psi_v(ax)$ for $x \in F_v$. Then

$$\gamma\left(\frac{1}{2}, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v}, \psi_v^a\right) = (\omega_1\omega_2\omega_3)^4(a)\gamma\left(\frac{1}{2}, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v}, \psi_v\right).$$

Therefore $\gamma(\Pi_v)$ is independent of the choice of ψ_v.

(3) If $\gamma(\Pi_v)^2 \neq 1$ for some v, then there exists a quaternion algebra D such that $\epsilon(D_v) \neq -\epsilon_K(-1)\gamma(\Pi_v)$ for all v (see Proposition 5.3).

When $K = F \times F$, Theorem 1.3 is nothing but Ichino's formula proved in [Ich08]. Ichino considers an étale cubic algebra over F. It should not be difficult to extend Theorem 1.3 to this case. The proof follows the same line as in the proof of [Ich08]. The global ingredient is the vanishing of incoherent Eisenstein series (see Proposition 5.1), which is combined with the seesaw identity. The assumption (Cusp) will be used to apply Proposition 2.11 at the last stage of the proof of Theorem 1.1. The local ingredient is the local functional equation of Garrett's zeta integral. Corollary 4.3 will relate the zeta integral to the sum of the two invariant trilinear forms. Then the local functional equation gives the relation stated in Proposition 4.7, from which Theorem 1.1 follows. Theorem 1.2 can be deduced from the unramified computation of the zeta integral.

Here is a short summary of the content of this paper. Section 2 describes the quaternary quadratic space V_D and studies theta lifts from GL_2 to $\mathrm{GO}(V_D)$. Section 3 constructs the local invariant trilinear forms. Section 4 relates those trilinear forms to (partial) zeta integrals and proves Theorems 1.1 and 1.2. Section 5 applies the seesaw machinery, following [HK91].

2. Twisted Shimizu correspondence

2.1. **Quaternary quadratic spaces.** Let D and K be a quaternion algebra and a quadratic extension over an arbitrary field F of characteristic zero. Fix an element $\delta \in F^\times \setminus F^{\times 2}$ so that $K = F(\sqrt{\delta})$. The main involution ι of D is uniquely determined by the conditions $x + x^\iota \in F$ and $xx^\iota \in F$ for every $x \in D$. The norm map $\mathrm{N}_{K/F} : K^\times \to F^\times$ is defined by $\mathrm{N}_{K/F}(k) = k\bar{k}$, where $\bar{\cdot}$ denotes the non-trivial automorphism of K over F.

Given a central simple algebra A over K, by an involution (anti-involution) of A, we mean an arbitrary F-linear automorphism (resp. anti-automorphism) of A of order 2. It is said to be of the second kind if its restriction to K coincides with $\bar{\cdot}$. Let $D_K = D \otimes_F K$ be a quaternion algebra over K. We K-linearly extend ι to an anti-involution of D_K, which is the main involution of D_K. An involution σ of D_K of the second kind can be defined by $\sigma(x \otimes k) = x \otimes \bar{k}$ for $x \in D$ and $k \in K$.

Involutions of $\mathrm{M}_n(K)$ of the first and second kind are defined by $x \mapsto {}^t x$ and $\varrho(x)_{ij} = \bar{x}_{ij}$ for $x = (x_{ij}) \in \mathrm{M}_n(K)$, where ${}^t x$ is the transpose of x. Put

$$\mathrm{Sym}_n(F) = \{b \in \mathrm{M}_n(F) \mid {}^t b = b\}, \qquad D^\circ = \{x \in D \mid x^\iota = -x\},$$

$$\mathrm{Her}_n(F) = \{\xi \in \mathrm{M}_n(K) \mid {}^t\varrho(\xi) = \xi\}, \qquad J_n = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}.$$

We define the symplectic similitude group by

$$\mathrm{GSp}_{2n} = \{g \in \mathrm{GL}_{2n} \mid gJ_n {}^t g = \nu_n(g) J_n,\ \nu_n(g) \in \mathbb{G}_m\}$$

and the similitude unitary group of a Hermitian matrix $\xi \in \mathrm{Her}_n(F)$ with $\det \xi \neq 0$ by

$$\mathrm{GU}(\xi) = \{g \in \mathrm{Res}_{K/F}\mathrm{GL}_n \mid g\xi {}^t\varrho(g) = \nu_\xi(g)\xi,\ \nu_\xi(g) \in \mathbb{G}_m\}.$$

The action ρ_1 of the group $D_K^\times$ on the subspace

$$V_D = \{x \in D_K \mid \sigma(x) = x^\iota\} = F \oplus \sqrt{\delta} D^\circ$$

is given by $\rho_1(\xi)x = \xi x \sigma(\xi)^\iota$. Define a quadratic form $(\ ,\ )_D$ on V_D by $(x,x)_D = xx^\iota$ for $x \in V_D$. The discriminant algebra of V_D is K.

The triplet $(D_K^\times, \rho_1, V_D)$ forms a prehomogeneous vector space. Note that

$$(\rho_1(\xi)x, \rho_1(\xi)x)_D = \mathrm{N}_{K/F}(\xi\xi^\iota)(x,x)_D$$

for $x \in V_D$ and $\xi \in D_K^\times$. Put

$$\mathcal{Y}_D = \{y \in V_D \mid (y,y)_D \neq 0\}.$$

Given $y \in \mathcal{Y}_D$, we can define an involution σ_y of D_K of the second kind by

$$\sigma_y(x) = \sigma(y^{-1}xy).$$

Lemma 2.1. *Let τ be an involution of D_K of the second kind. Then there exists $y \in \mathcal{Y}_D$ such that $\tau = \sigma_y$.*

Proof. The Skolem-Noether theorem implies that $\tau(x)^\iota = \tau(x^\iota)$ for all $x \in D_K$. Thus we can define two anti-involutions σ_0 and τ_0 of D_K of the second kind by $\sigma_0(x) = \sigma(x^\iota)$ and $\tau_0(x) = \tau(x^\iota)$ for $x \in D_K$. Lemma 2.10 of [PR94] gives an element $y \in D_K^\times$ such that $\tau_0(x) = y^{-1}\sigma_0(x)y$ and $\sigma_0(y) = y$. □

Let $D_y := \{x \in D_K \mid \sigma_y(x) = x\}$ be an F-subalgebra of D_K.

Lemma 2.2. *D_y is a quaternion algebra over F such that $D_y \otimes K = D_K$.*

Proof. It is evident that D_y has dimension 4 over F and $D_y \otimes K = D_K$. Thus D_y is central over F. If D_y has a non-trivial two-sided ideal, then so does D_K. □

Let $\mathrm{GO}(V_D)$ denote the orthogonal similitude group of V_D defined by

$$\mathrm{GO}(V_D) = \{h \in \mathrm{GL}(V_D) \mid (hx, hy)_D = \boldsymbol{\nu}(h)(x,y)_D,\ \boldsymbol{\nu}(h) \in \mathbb{G}_m\}.$$

The subgroup $\mathrm{GSO}(V_D)$ of $\mathrm{GO}(V_D)$ consists of the elements h such that $\det h = \boldsymbol{\nu}(h)^2$. We view $D_K^\times$ as a subgroup of $\mathrm{GSO}(V_D)$ via ρ_1. There is an exact sequence

$$1 \to K^\times \xrightarrow{\rho_0} (F^\times \times D_K^\times) \rtimes \langle \mathbf{t} \rangle \xrightarrow{\rho} \mathrm{GO}(V_D) \to 1, \tag{2.1}$$

where

$$\rho_0(c) = (\mathrm{N}_{K/F}(c), c^{-1}), \qquad \rho(a,\xi)v = a\rho_1(\xi)v, \qquad \rho(\mathbf{t})v = v^\iota$$

for $v \in V_D$, and $\mathbf{t}$ acts on $D_K^\times$ by $x \mapsto \sigma(x)$. Observe that

$$\boldsymbol{\nu}(\rho(a,\xi)) = a^2\mathrm{N}_{K/F}(\xi\xi^\iota).$$

Given $a \in F^\times$ and a quadratic space $(V, (\ ,\ ))$, we write V^a for the space V quipped with the quadratic form $a(\ ,\ )$. Put

$$V_{D_y} = \{x \in D_K \mid \sigma_y(x) = x^\iota\}, \qquad \mathcal{V}_y = \{\xi \in D_K^\times \mid \xi\sigma_y(\xi)^\iota \in F^\times\}.$$

Equip V_{D_y} with a quadratic form defined by $(x,x)_{D_y} = xx^\iota$ for $x \in V_{D_y}$.

The following lemma is straightforward to prove.

Lemma 2.3. *Notation being as above, we have* $V_{D_y} = V_D y^{-1}$. *Put* $a = (y, y)_D$. *Then the map* $x \mapsto xy$ *for* $x \in V_{D_y}$ *gives an isomorphism of* $V_{D_y}^a$ *onto* V_D.

When $D = \mathrm{M}_2(F)$, we write $V_+ = V_D$ and $(\ ,\ )_+ = (\ ,\ )_D$. In this case

$$x^\iota = J_1 \, {}^t x J_1^{-1}, \qquad \sigma = \varrho, \qquad V_+ = \sqrt{\delta} J_1 \cdot \mathrm{Her}_2(F)$$

for $x \in D_K$. Note that $(\sqrt{\delta} J_1 \xi, \sqrt{\delta} J_1 \xi)_+ = \delta \det \xi$ for $\xi \in \mathrm{Her}_2(F)$.

2.2. **Quaternary quadratic spaces over local fields.** Let F be a local field of characteristic zero. Then $D_K \simeq \mathrm{M}_2(K)$. We denote by $\epsilon(D)$ the Hasse invariant of D and by ϵ_K the quadratic character of $F^\times$ whose kernel is $\mathrm{N}_{K/F}(K^\times)$. If $\epsilon(D) = -1$, then V_+ and V_D have opposite Hasse invariants.

Remark 2.4. The quadratic space V_+ is isomorphic to the orthogonal sum of the norm form on K with a split binary quadratic space as a quadratic space. Take $\alpha \in F^\times$ so that $\epsilon_K(\alpha) = \epsilon(D)$. Then $V_D \simeq V_+^\alpha$.

Proposition 2.5. *Let* $y \in \mathcal{Y}_D$.

(1) Fix an element $y_- \in \mathcal{Y}_D$ *with* $\epsilon_K(\det y_-) = -1$. *Then*

$$\mathcal{Y}_D = \rho_1(\mathrm{GL}_2(K))\mathbf{1}_2 \sqcup \rho_1(\mathrm{GL}_2(K)) y_-.$$

(2) $\epsilon(D_y) = \epsilon_K((y, y)_D)\epsilon(D)$.

(3) $\mathcal{V}_y = K^\times D_y^\times$.

Proof. If $y' \in \mathcal{Y}_{D_y}$, then

$$y'y \in \mathcal{Y}_D, \qquad \sigma_{y'y} = (\sigma_y)_{y'}, \qquad D_{y'y} = (D_y)_{y'}.$$

Thanks to Lemma 2.1, we may assume that $D = \mathrm{M}_2(F)$ and $\sigma = \varrho$. Put

$$\mathcal{Y} := \mathcal{Y}_{\mathrm{M}_2(F)} = \{\sqrt{\delta} J_1 \cdot \xi \mid \xi \in \mathrm{Her}_2(F),\ \det \xi \neq 0\}.$$

The group $\mathrm{GL}_2(K)$ acts on $\mathcal{Y}$ by $\rho_1(g)y = gy\varrho(g)^\iota = gyJ_1 \, {}^t\varrho(g) J_1^{-1}$ for $g \in \mathrm{GL}_2(K)$ and $y \in \mathcal{Y}$. Note that $\rho_1(g)(\xi \cdot \sqrt{\delta} J_1) = g\xi \, {}^t\varrho(g) \cdot \sqrt{\delta} J_1$. Since there are two equivalence classes of non-degenerate Hermitian matrices ξ of size 2 classified by the sign $\epsilon_K(-\det \xi)$, one can deduce (1).

We write $y = \sqrt{\delta} J_1 \cdot {}^t\xi^{-1}$ with $\xi \in \mathrm{Her}_2(F)$. Let $x \in \mathrm{M}_2(F)_y$. Then

$$x = \varrho(y^{-1}xy) = \xi J_1^{-1} \varrho(x) J_1 \xi^{-1} = \xi \, {}^t\varrho(x)^\iota \xi^{-1}, \qquad x\xi \, {}^t\varrho(x) = \overline{\det x} \cdot \xi.$$

It is well-known that

$$\mathrm{GU}(\xi) \simeq (D_\xi^\times \times K^\times)/F^\times,$$

where D_ξ is a quaternion algebra with $\epsilon(D_\xi) = \epsilon_K(-\det \xi) = \epsilon_K(\det y)$. Now (2) follows from the observation

$$\mathrm{M}_2(F)_y \cap \mathrm{GL}_2(K) = \{g \in \mathrm{GU}(\xi) \mid \lambda_\xi(g) = \det g\} \simeq D_\xi^\times.$$

Clearly, $K^\times D_y^\times \subset \mathcal{V}_y$. We shall prove the reverse inclusion. Let $\xi \in \mathcal{V}_y$. Put $a := \xi \sigma_y(\xi)^\iota \in F^\times$. Since σ_y has order 2 and acts on K non-trivially, we get $a = \sigma_y(a) = \sigma_y(\xi)\xi^\iota$ and $a^2 = \mathrm{N}_{K/F}(\det\xi)$. We can therefore take $k \in K^\times$ so that $\det\xi = a\bar{k}k^{-1}$. Then since

$$\xi\xi^\iota = a\bar{k}k^{-1} = \xi\sigma_y(\xi)^\iota\bar{k}k^{-1},$$

we have $\xi k = \sigma_y(\xi k)$, which proves (3). □

2.3. **The Weil representation for similitudes.** Given $a \in \mathrm{GL}_n$, $b \in \mathrm{Sym}_n$ and a scalar $t \in \mathbb{G}_m$, we put

$$\mathbf{m}(a) = \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix}, \qquad \mathbf{n}(b) = \begin{pmatrix} \mathbf{1}_n & b \\ & \mathbf{1}_n \end{pmatrix}, \qquad \mathbf{d}(t) = \begin{pmatrix} \mathbf{1}_n & \\ & t \cdot \mathbf{1}_n \end{pmatrix}.$$

Let $P_n = MN$ be the Siegel parabolic subgroup of GSp_{2n} given by

$$M = \{\mathbf{m}(a)\mathbf{d}(t) \mid a \in \mathrm{GL}_n,\ t \in \mathbb{G}_m\}, \qquad N = \{\mathbf{n}(b) \mid b \in \mathrm{Sym}_n\}.$$

We denote the kernels of the similitude characters $\nu_n : \mathrm{GSp}_{2n} \to \mathbb{G}_m$ and $\boldsymbol{\nu} : \mathrm{GO}(V_D) \to \mathbb{G}_m$ by Sp_{2n} and $\mathrm{O}(V_D)$, and the centers of GSp_{2n} and $\mathrm{GO}(V_D)$ by Z_n and Z_D. Note that Z_n and Z_D are isomorphic to $\mathbb{G}_m$.

Let F be a number field with adèle ring $\mathbb{A}$ and ϵ_K the quadratic Hecke character corresponding to a quadratic extension K/F via class field theory. Fix a non-trivial additive character $\psi = \prod_v \psi_v$ of $\mathbb{A}/F$. Let $\Omega^n_{D,\psi} = \otimes_v \Omega^n_{D_v,\psi_v}$ denote the Weil representation of $\mathrm{Sp}_{2n}(\mathbb{A}) \times \mathrm{O}(V_D, \mathbb{A})$ with respect to ψ on the Schwartz space $\mathcal{S}(V_D^n(\mathbb{A}))$ with

$$\begin{aligned} &\Omega^n_{D,\psi}(\mathbf{m}(a))\Phi(x) = \epsilon_K(\det a)|\det a|^2\Phi(xa), && a \in \mathrm{GL}_n(\mathbb{A}), \\ &\Omega^n_{D,\psi}(\mathbf{n}(b))\Phi(x) = \psi(\mathrm{tr}(b(x,x)_D))\Phi(x), && b \in \mathrm{Sym}_n(\mathbb{A}), \\ &\Omega^n_{D,\psi}(h)\Phi(x) = \Phi(h^{-1}x), && h \in \mathrm{O}(V_D, \mathbb{A}), \end{aligned} \tag{2.2}$$

where $(x,x)_D = ((x_i, x_j)_D) \in \mathrm{Sym}_n(\mathbb{A})$.

On the orthogonal similitude group $\mathrm{GO}(V_D, \mathbb{A})$ we can extend $\Omega^n_{D,\psi}$ by

$$L(h)\Phi(x) = |\boldsymbol{\nu}(h)|^{-n}\Phi(h^{-1}x).$$

We use it to extend $\Omega^n_{D,\psi}$ to a representation of the group

$$R_n = \{(h,g) \in \mathrm{GO}(V_D) \times \mathrm{GSp}_{2n} \mid \nu_n(g) = \boldsymbol{\nu}(h)\}.$$

Since $L(h)\Omega^n_{D,\psi}(g)L(h)^{-1} = \Omega^n_{D,\psi}(\mathbf{d}(t)g\mathbf{d}(t)^{-1})$ with $t = \boldsymbol{\nu}(h)$ for $g \in \mathrm{Sp}_{2n}(\mathbb{A})$ and $h \in \mathrm{GO}(V_D, \mathbb{A})$, one obtains a representation of the semidirect product $\mathrm{GO}(V_D, \mathbb{A}) \ltimes \mathrm{Sp}_{2n}(\mathbb{A})$ on $\mathcal{S}(V_D^n(\mathbb{A}))$. By composition with the isomorphism $(h,g) \to (h, \mathbf{d}(\boldsymbol{\nu}(h))^{-1}g)$ from R_n onto $\mathrm{GO}(V_D) \ltimes \mathrm{Sp}_{2n}$, we get the representation of $R_n(\mathbb{A})$ on $\mathcal{S}(V_D^n(\mathbb{A}))$, which we denote also by $\Omega^n_{D,\psi}$.

Remark 2.6. Note that for $z \in \mathbb{A}^\times$ and $\Phi \in \mathcal{S}(V_D^n(\mathbb{A}))$

$$\Omega^n_{D,\psi}(z,z)\Phi = \epsilon_K(z)^n\Phi.$$

We can form the theta series as a function on $R_n(F)\backslash R_n(\mathbb{A})$ defined by

$$\Theta(h,g;\Phi) = \sum_{x \in V_D^n(F)} \Omega^n_{D,\psi}(h,g)\Phi(x).$$

Definition 2.7. Let $\mathbb{B}$ be the open subgroup of $\mathbb{A}^\times$ which consists of idèles $\boldsymbol{\nu}(h)$ with $h \in \mathrm{GO}(V_D, \mathbb{A})$. Let S_D^K be the set of real places of F at which either K or D is not split. When $v \in S_D^K$, the subgroup $\mathcal{B}_v$ consists of positive real numbers in $F_v^\times$. If $v \notin S_D^K$, then we set $\mathcal{B}_v = \mathrm{N}_{K_v/F_v}(K_v^\times)$. Put

$$\begin{aligned}\mathrm{GSp}_{2n}(\mathbb{A})^\star &= \{g \in \mathrm{GSp}_{2n}(\mathbb{A}) \mid \nu_n(g) \in \mathbb{B}\},\\ \mathrm{GSp}_{2n}(F)^\star &= \mathrm{GSp}_{2n}(F) \cap \mathrm{GSp}_{2n}(\mathbb{A})^\star,\\ \mathrm{GSp}_{2n}(F_v)^\star &= \{g \in \mathrm{GSp}_{2n}(F_v) \mid \nu_n(g) \in \mathcal{B}_v\}.\end{aligned}$$

2.4. **The quadratic base change as a theta lift.** Let $n = 1$. Then $\mathrm{GSp}_2 \simeq \mathrm{GL}_2$. We start with a quadratic extension K/F of non-archimedean local fields of characteristic zero. Fix a non-trivial additive character ψ on F and a quaternion algebra D over F. We will abbreviate $\Omega^1_{D,\psi} = \Omega_\psi^{\epsilon(D)}$ to denote the local Weil representation.

Recall the subgroup

$$\mathrm{GL}_2(F)^\star = \mathrm{GSp}_2(F)^\star = \{g \in \mathrm{GL}_2(F) \mid \epsilon_K(\det g) = 1\}.$$

Let $\pi^\star$ be an infinite-dimensional irreducible admissible representation of $\mathrm{GL}_2(F)^\star$. The maximal $(\pi^\star)^\vee$-isotypic quotient of $\text{c-ind}_{R_1}^{\mathrm{GO}(V)\times\mathrm{GL}_2(F)^\star}\Omega_\psi^{\epsilon(D)}$ is of the form $(\pi^\star)^\vee \boxtimes \Theta_K^D(\pi^\star)$, where $(\pi^\star)^\vee$ is the contragredient representation of $\pi^\star$ and $\Theta_K^D(\pi^\star)$ is a (possibly zero) smooth representation of $\mathrm{GO}(V_D)$.

Let π be a generic irreducible admissible representation of $\mathrm{GL}_2(F)$ of central character ω. We write π_K for the base change of π to $\mathrm{GL}_2(K)$.

Definition 2.8. When $\pi|_{\mathrm{GL}_2(F)^\star}$ is reducible, Lemma 4.1 of [GI11] allows us to write $\pi|_{\mathrm{GL}_2(F)^\star} = \pi^+ \oplus \pi^-$, where $\pi^\pm$ are irreducible representations of $\mathrm{GL}_2(F)^\star$ such that

$$\Theta_K^D(\pi^+) \neq 0, \qquad\qquad \Theta_K^D(\pi^-) = 0.$$

We set

$$\Theta_K^D(\pi) = \begin{cases}\Theta_K^D(\pi^+) & \text{if } \pi|_{\mathrm{GL}_2(F)^\star} \text{ is reducible,}\\ \Theta_K^D(\pi|_{\mathrm{GL}_2(F)^\star}) & \text{if } \pi|_{\mathrm{GL}_2(F)^\star} \text{ is irreducible.}\end{cases}$$

Proposition 2.9 ([Lu17]). $\Theta_K^D(\pi)$ *is nonzero, irreducible and*

$$\Theta_K^D(\pi)|_{D_K^\times \times F^\times} \simeq \pi_K \boxtimes \omega\epsilon_K.$$

Proof. See Section 3 of [Lu17] (cf. (2.1)). □

We switch to the global setting. Thus F is a number field and π an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ with central character ω. For a technical reason we assume that all the archimedean places of F are split in K. Given a cusp form $f \in \pi$ and $\varphi \in \mathcal{S}(V_D(\mathbb{A}))$, we define an automorphic form on $\mathrm{GO}(V_D, \mathbb{A})$ by

$$\theta(\xi; f, \varphi) = \int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})} f(gg')\Theta(\xi, gg'; \varphi)\,\mathrm{d}g,$$

where we choose $g' \in \mathrm{GL}_2(\mathbb{A})^\star$ so that $\det g' = \boldsymbol{\nu}(\xi)$. Here, $\mathrm{d}g = \prod_v \mathrm{d}g_v$ is the Tamagawa measure on $\mathrm{SL}_2(\mathbb{A})$. Let $\theta_K^D(\pi)$ denote the automorphic representation of $\mathrm{GO}(V_D, \mathbb{A})$ generated by $\theta(\xi; f, \varphi)$, as $\varphi \in \mathcal{S}(V_D(\mathbb{A}))$ and $f \in \pi$ vary.

We denote the base change of π to $\mathrm{GL}_2(\mathbb{K})$ by π_K. We denote the Jacquet-Langlands lift of π_K to $D_K^\times(\mathbb{A})$ by π_K^D if it exists. By the following result π_K^D can be extended to a representation of $\mathrm{GO}(V_D, \mathbb{A})$.

Proposition 2.10. *Assume that π_K is cuspidal. The space $\theta_K^D(\pi)$ is not zero precisely when π_K^D exists. In this case*

$$\theta_K^D(\pi)|_{D_K^\times(\mathbb{A}) \times \mathbb{A}^\times} \simeq \pi_K^D \boxtimes \omega\epsilon_K.$$

Proof. This is due, in essence, to [Shi72]. The standard L-function of $\pi|_{\mathrm{SL}_2(\mathbb{A})}$ twisted by ϵ_K is $L(s, \mathrm{Ad}(\pi) \otimes \epsilon_K)$ and is holomorphic and not zero at $s = 1$ by assumption (cf. Remark 1.4(1)). Theorem 11.6 of [GQT14] applied to the restriction of $\theta(\xi; f, \varphi)$ to $\mathrm{SO}(V_D, \mathbb{A})$ implies that the global theta lift $\theta_K^D(\pi)$ is not zero if the local theta lift of π_v to $\mathrm{GO}(V_{D_v})$ is not zero for all v. □

2.5. **Factorization of the Flicker-Rallis period.** Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ whose base change π_K is cuspidal. We define an element

$$B \in \mathrm{Hom}_{D^\times(\mathbb{A})}(\pi_K^D, (\omega\epsilon_K) \circ \mathrm{N}_{D/F})$$

by the period integral

$$B(\phi) = \int_{\mathbb{A}^\times D^\times(F) \backslash D^\times(\mathbb{A})} \phi(h)(\omega^{-1}\epsilon_K)(hh^\iota)\,\mathrm{d}h,$$

where $\mathrm{d}h$ is the Tamagawa measure on $\mathbb{A}^\times \backslash D^\times(\mathbb{A})$.

Define the Whittaker function of $f \in \pi$ with respect to $\bar{\psi}$ by

$$W_f(g) = \int_{F \backslash \mathbb{A}} f(\mathbf{n}(b)g)\psi(b)\,\mathrm{d}b,$$

where $\mathrm{d}b = \prod_v \mathrm{d}b_v$ is the Tamagawa measure on $\mathbb{A}$. Assume that $W_f(g) = \prod_v W_v(g_v)$, where $W_v(\mathbf{1}_2) = 1$ for almost all v. Define a map $\tilde{B}_v : \pi_v \otimes \mathcal{S}(V_{D_v}) \to \mathbb{C}$ by

$$\tilde{B}_v(W_v, \varphi_v) = \int_{U(F_v) \backslash \mathrm{SL}_2(F_v)} W_v(\dot{g}_v)\Omega_{\psi_v}^{\epsilon(D_v)}(\dot{g}_v)\varphi_v(1)\,\mathrm{d}\dot{g}_v$$

for each place v of F, where $U = \{\mathbf{n}(b) \mid b \in \mathbb{G}_a\}$ and $\mathrm{d}\dot{g}_v$ is the quotient measure of $\mathrm{d}g_v$ by $\mathrm{d}b_v$. Since there exists $\varphi_0 \in \mathcal{S}(F_v)$ and for $\epsilon > 0$ there exists $\varphi_\epsilon \in \mathcal{S}(F_v)$ such that

$$|\Omega_{\psi_v}^{\epsilon(D_v)}(\mathbf{m}(a)k)\varphi_v(1)| \leq |a|^2\varphi_0(a), \qquad |W_v(m(a)k)| \leq |a|^\epsilon\varphi_\epsilon(a^2)$$

for $a \in F_v^\times$ and $k \in \mathrm{SL}_2(\mathfrak{o}_{F_v})$ (cf. p. 298 of [Ich08]), the integral converges absolutely.

Let $S_{f,\varphi}$ be a finite set of places of F including all archimedean places so that for $v\notin S_{f,\varphi}$,

- ϵ_{K_v} is unramified,
- ψ_v is trivial on $\mathfrak{o}_{F_v}$ but non-trivial on $\mathfrak{p}_v^{-1}$,
- $W_v(\mathrm{GL}_2(\mathfrak{o}_{F_v}))=1$,
- $\epsilon(D_v)=1$ and φ_v is the characteristic function of $V_{D_v}\cap\mathrm{M}_2(\mathfrak{o}_{K_v})$,
- $\mathrm{vol}(\mathrm{SL}_2(\mathfrak{o}_v),\mathrm{d}g_v)=1$.

Here $\mathfrak{o}_{F_v}$ and $\mathfrak{o}_{K_v}$ are the maximal compact subrings of F_v and K_v, and $\mathfrak{p}_v$ is the maximal ideal of $\mathfrak{o}_{F_v}$.

Proposition 2.11. *(1) If $v\notin S_{f,\varphi}$, then $\tilde{B}_v(W_v,\varphi_v)=\frac{L(1,\mathrm{Ad}(\pi_v)\otimes\epsilon_{K_v})}{\zeta_{F_v}(2)}$.*
(2) If π_K is cuspidal, then

$$B(\theta(f,\varphi))=2\zeta_F^S(2)^{-1}L^S(1,\mathrm{Ad}(\pi)\otimes\epsilon_K)\prod_{v\in S_{f,\varphi}}\tilde{B}_v(W_v,\varphi_v).$$

Remark 2.12. Proposition 5 of [Wal85] deals with the case $K=F\times F$.

Let $v\notin S_{f,\varphi}$. Fix a prime element ϖ_v of $\mathfrak{o}_{F_v}$. Then

$$\tilde{B}_v(W_v,\varphi_v)=\sum_{i=0}^{\infty}W_v(\mathbf{m}(\varpi_v^i))\epsilon_{K_v}(\varpi_v^i).$$

Since the Shintani formula (cf. [Wal85, p. 190]) gives

$$W_v(\mathbf{m}(\varpi_v^i))=(\alpha_v\beta_v)^{-i}W_v\left(\begin{pmatrix}\varpi_v^{2i} & \\ & 1\end{pmatrix}\right)=(\alpha_v\beta_v)^{-i}q_v^{-i}\frac{\alpha_v^{2i+1}-\beta_v^{2i+1}}{\alpha_v-\beta_v},$$

where $\{\alpha_v,\beta_v\}$ is the Satake parameter of π_v and q_v is the cardinality of the residue field of $\mathfrak{o}_{F_v}$, we get (1) by a simple calculation.

When D is not split, one can use the Siegel-Weil formula to prove the formula (2) as in the proof of Proposition 2.3 of [YZZ13]. The rest of this section is devoted to proving Proposition 2.11(2) for $D=\mathrm{M}_2(F)$ and $\sigma=\varrho$.

Recall that $K=F(\sqrt{\delta})$. Define an additive character ψ_K^δ on $\mathbb{K}$, which is trivial on $K+\mathbb{A}$, by $\psi_K^\delta(k)=\psi\left(\mathrm{T}_{K/F}\left(\frac{k}{\sqrt{\delta}}\right)\right)$ for $k\in\mathbb{K}$, where $\mathrm{T}_{K/F}$ is the trace map from $\mathbb{K}$ to $\mathbb{A}$. We define the Whittaker function of $\phi\in\pi_K$ with respect to ψ_K^δ by

$$W_\phi(g)=\int_{K\backslash\mathbb{K}}\phi(\mathbf{n}(k)g)\overline{\psi_K^\delta(k)}\,\mathrm{d}k,$$

where $\mathrm{d}k$ is the Tamagawa measure on $\mathbb{K}$. Let $\mathrm{d}a_v$ and $\mathrm{d}c_v$ be the self-dual Haar measures of F_v with respect to ψ_v. Put $\mathrm{d}^\times a_v=\zeta_{F_v}(1)\frac{\mathrm{d}a_v}{|a_v|}$. For $a_v\in F_v^\times$ we put $\mathbf{t}(a_v)=\begin{pmatrix}a_v & 0\\ 0 & 1\end{pmatrix}\in\mathrm{GL}_2(F_v)$.

Proposition 2.13. *(1) There exists $\phi\in\pi_K$ such that $B(\phi)\neq 0$.*

(2) Let $\phi \in \pi_K$ be factorizable, i.e., $W_\phi = \otimes_v W_{\phi_v}$. Then

$$B(\phi) = 2L(1, \mathrm{Ad}(\pi) \otimes \epsilon_K) \prod_v \frac{\beta_v(W_{\phi_v})}{\zeta_{F_v}(1) L(1, \mathrm{Ad}(\pi_v) \otimes \epsilon_{K_v})},$$

where

$$\beta_v(W_{\phi_v}) = \int_{F_v^\times} W_{\phi_v}(\mathbf{t}(a_v))(\omega_v^{-1}\epsilon_{K_v})(a_v)\, \mathrm{d}^\times a_v.$$

Proof. We extend $\omega^{-1}\epsilon_K$ to a character γ of $\mathbb{K}^\times/K^\times$. Put $\sigma = \pi_K \otimes \gamma$. The Asai L-function of σ is $\zeta_F(s)L(s, \mathrm{Ad}(\pi)\otimes\epsilon_K)$ and so by Remark 1.4(1), it has a pole at $s = 1$, which proves (1) (see [FZ95]). Put $\mathbb{A}^1 = \{a \in \mathbb{A}^\times \mid |a| = 1\}$. Since the volume of $F^\times\backslash\mathbb{A}^1$ with respect to $\prod_v \mathrm{d}^\times a_v$ is the residue of $\zeta_F(s)$ at $s = 1$, Proposition 3.2 of [Zha14] includes (2). □

Once we establish the identity

$$\beta_v(W_{\phi_v}) = \zeta_{F_v}(1)\tilde{B}_v(W_v, \varphi_v),$$

Proposition 2.11(2) follows from Proposition 2.13(2). The mixed model of $\Omega^+_{\psi_v}$ is realized on $\mathcal{S}(K_v \oplus F_v^2)$. The intertwining map $I : \mathcal{S}(V^+(F_v)) \to \mathcal{S}(K_v \oplus F_v^2)$ is given by a partial Fourier transform

$$I(\varphi_v)(k; a, b) = \int_{F_v} \psi_v(c_v a)\varphi_v\left(\begin{pmatrix} k & \sqrt{\delta}c_v \\ \frac{b}{\sqrt{\delta}} & \bar{k} \end{pmatrix}\right)\mathrm{d}c_v$$

(see §5.2 of [KR94]). Let $t \in F_v^\times$. Since $\boldsymbol{\nu}(\mathbf{d}(t)) = \det(t \cdot \mathbf{1}_2) = t^2$, Remark 2.6 gives

$$\Omega^+_{\psi_v}(\mathbf{d}(t), t \cdot \mathbf{1}_2)\varphi_v = \epsilon_{K_v}(t)\Omega^+_{\psi_v}(t^{-1} \cdot \mathbf{d}(t), \mathbf{1}_2)\varphi_v,$$

from which it follows that

$$\begin{aligned} &I(\Omega^+_{\psi_v}(\mathbf{d}(t), t \cdot \mathbf{1}_2)\varphi_v)(k; a, b) \\ =&\epsilon_{K_v}(t)\int_{F_v} \psi_v(c_v a)\varphi_v\left(t\mathbf{d}(t^{-1})\begin{pmatrix} k & \sqrt{\delta}c_v \\ \frac{b}{\sqrt{\delta}} & \bar{k} \end{pmatrix}\mathbf{d}(t^{-1})^\iota\right)\mathrm{d}c_v \\ =&\epsilon_{K_v}(t)|t|^{-1}I(\varphi_v)(k; t^{-1}a, t^{-1}b). \end{aligned}$$

Let $f \in \pi$ and $\varphi \in \mathcal{S}(V^+(\mathbb{A}))$ be factorizable. Put $\phi = \theta(f, \varphi)$. The Whittaker function of ϕ with respect to ψ_K^δ is given by

$$W_\phi(\xi) = \int_{U(\mathbb{A})\backslash\mathrm{SL}_2(\mathbb{A})} W_f(\dot{g})I(\Omega^+_\psi(\xi, \dot{g}g')\varphi)(1; 1, 0)\,\mathrm{d}\dot{g} = \prod_v W_{\phi_v}(\xi_v)$$

for $\xi \in \mathrm{GL}_2(\mathbb{K})$, where $\det g' = \mathrm{N}_{K/F}(\det\xi)$ (see §5.1 of [Lu17]). We have

$$\begin{aligned} W_{\phi_v}(\mathbf{t}(t)) &= \omega_v(t)W_{\phi_v}(\mathbf{d}(t^{-1})) \\ &= (\omega_v\epsilon_{K_v})(t)|t|\int_{U(F_v)\backslash\mathrm{SL}_2(F_v)} W_v(\dot{g}_v)I(\Omega^+_{\psi_v}(\dot{g}_v)\varphi_v)(1; t, 0)\,\mathrm{d}\dot{g}_v. \end{aligned}$$

By the Fourier inversion formula we get

$$\begin{aligned}\beta_v(W_{\phi_v}) &= \int_{F_v^\times} W_{\phi_v}(\mathbf{t}(t_v))(\omega_v^{-1}\epsilon_{K_v})(t_v)\,\mathrm{d}^\times t_v \\ &= \zeta_{F_v}(1)\int_{U(F_v)\backslash \mathrm{SL}_2(F_v)} W_v(\dot{g}_v)\Omega^+_{\psi_v}(\dot{g}_v)\varphi_v(\mathbf{1}_2)\,\mathrm{d}\dot{g}_v\end{aligned}$$

as claimed.

3. Local trilinear forms

3.1. **Flicker-Rallis functionals.** In this and the next section we fix an inert place v of F and suppress it from the notation. Thus $F = F_v$ is a non-archimedean local field of characteristic zero, K a quadratic extension of F, D a quaternion algebra over F, ψ a fixed non-trivial additive character of F, and ϵ_K the quadratic character of $F^\times$ whose kernel is $\mathrm{N}_{K/F}(K^\times)$. We denote by $\mathrm{N}_{D/F} : D^\times \to F^\times$ the reduced norm and by $\tau_{D/F} : D \to F$ the reduced trace. Let $\boldsymbol{\alpha}_F(z) = |z|$ denote the normalized absolute value of $z \in F^\times$.

Recall that $D_K = D \otimes K \simeq \mathrm{M}_2(K)$. The main involution of D induces an anti-involution ι of D_K of the first kind. Let σ be the involution of D_K of the second kind such that $D = \{x \in D_K \mid \sigma(x) = x\}$. Equip $V_D = \{x \in D_K \mid \sigma(x) = x^\iota\}$ with a quadratic form defined by $(x,x)_D = xx^\iota$. The discriminant character of V_D is ϵ_K. The morphisms $\rho_1 : D_K^\times \to \mathrm{GSO}(V_D)$ and $\boldsymbol{\nu} : D_K^\times \to F^\times$ are given by $\rho_1(\xi)x = \xi x \sigma(\xi)^\iota$ (see (2.1)) and $\boldsymbol{\nu}(\xi) = \mathrm{N}_{K/F}(\xi\xi^\iota)$ for $x \in V_D$ and $\xi \in D_K^\times$.

Let π be an irreducible unitary admissible infinite-dimensional representation of $\mathrm{GL}_2(F)$ whose central character is ω. Given $a \in F^\times$, we define an additive character ψ^a on F by setting $\psi^a(b) = \psi(ab)$ for $b \in F$. We denote by π_K the base change lift of π to $\mathrm{GL}_2(K)$, by $W^{\bar{\psi}^a}(\pi)$ the Whittaker model of π with respect to $\bar{\psi}^a$, and by $\lambda(\pi)$ the real number defined by

$$\lambda(\pi) = \begin{cases} 0 & \text{if } \pi \text{ is tempered,} \\ |\lambda| & \text{if } \pi = \mathrm{Ind}^{\mathrm{GL}_2(F)}_{P_1(F)}(\chi\boldsymbol{\alpha}_F^\lambda \boxtimes \omega\chi^{-1}\boldsymbol{\alpha}_F^{-\lambda}),\end{cases}$$

where $\lambda \in \mathbb{R}$ and χ is a unitary character of $F^\times$.

Given $W \in W^{\bar{\psi}}(\pi)$, we define $W^\alpha \in W^{\bar{\psi}^\alpha}(\pi)$ by

$$W^\alpha(g) = W(\mathbf{d}(\alpha)^{-1}g).$$

Fix $y \in \mathcal{Y}_D$. Put $\alpha = (y,y)_D$. For $\varphi \in \mathcal{S}(V_D)$ and $\xi \in \mathrm{GO}(V_D)$ we put

$$\tilde{B}_y(\xi; W, \varphi) = \int_{U\backslash \mathrm{SL}_2(F)} W^\alpha(\dot{g}\mathbf{d}(\boldsymbol{\nu}(\xi)))\Omega^{\epsilon(D)}_\psi(\xi, \dot{g}\mathbf{d}(\boldsymbol{\nu}(\xi)))\varphi(y)\,\mathrm{d}\dot{g}.$$

One can see that this integral converges absolutely, likewise for $\tilde{B}$.

Remark 3.1. Taking Lemma 2.3 into account, we define $\varphi_y \in \mathcal{S}(V_{D_y})$ by $\varphi_y(x) = \varphi(xy)$ for $x \in V_{D_y}$. It is easy to see that for $x \in V_{D_y}$

$$\Omega_\psi^{\epsilon(D)}(h,g)\varphi(xy) = \Omega_{\psi^\alpha}^{\epsilon(D_y)}(h,g)\varphi_y(x).$$

Lemma 3.2. *For $k \in K^\times$, $h \in D_y^\times$ and $\xi \in \mathrm{GO}(V_D)$ we have*

$$\tilde{B}_y(kh\xi; W, \varphi) = (\omega\epsilon_K)(k\bar{k}hh^\iota)\tilde{B}_y(\xi; W, \varphi).$$

Moreover, if $\xi \in D_K^\times$, then

$$\tilde{B}_y(\xi; W, \varphi) = \int_{U\backslash \mathrm{SL}_2(F)} W^\alpha(\mathbf{d}(\boldsymbol{\nu}(\xi))\dot{g})\Omega_{\psi^\alpha}^{\epsilon(D_y)}(\dot{g})\varphi_y(\xi^{-1}\sigma_y(\xi^{-1})^\iota)\,\mathrm{d}\dot{g}.$$

Proof. The first part can be derived from (2.2) or Proposition 2.9. Changing the variable $g \mapsto \mathbf{d}(\boldsymbol{\nu}(\xi))g\mathbf{d}(\boldsymbol{\nu}(\xi)^{-1})$, we get

$$\tilde{B}_y(\xi; W, \varphi) = \int_{U\backslash \mathrm{SL}_2(F)} W^\alpha(\mathbf{d}(\boldsymbol{\nu}(\xi))\dot{g})\Omega_\psi^{\epsilon(D)}(\rho_1(\xi), \mathbf{d}(\boldsymbol{\nu}(\xi))\dot{g})\varphi(y)|\boldsymbol{\nu}(\xi)|\,\mathrm{d}\dot{g}.$$

For $g \in \mathrm{SL}_2(F)$ and $\xi \in D_K^\times$ we have

$$\Omega_\psi^{\epsilon(D)}(\rho_1(\xi), \mathbf{d}(\boldsymbol{\nu}(\xi))g)\varphi(y) = |\boldsymbol{\nu}(\xi)|^{-1}\Omega_\psi^{\epsilon(D)}(g)\varphi(\xi^{-1}y\sigma(\xi^{-1})^\iota).$$

Since $y\sigma(x)^\iota = \sigma(x\sigma(y^\iota))^\iota = \sigma(xy)^\iota = (\sigma(y)\sigma_y(x))^\iota = \sigma_y(x)^\iota y$ for $x \in D_K$, we get the stated expression. □

By Proposition 2.9 there exists an equivariant surjective map

$$\theta_y : W^{\bar{\psi}}(\pi)|_{\mathrm{SL}_2(F)} \otimes \mathcal{S}(V_D) \to \Theta_K^{D_y}(\pi).$$

Lemma 3.2 gives rise to the following functional B_y.

Proposition 3.3. *There is $B_y \in \mathrm{Hom}_{D_y^\times}(\Theta_K^{D_y}(\pi), (\omega\epsilon_K)\circ \mathrm{N}_{D_y/F})$ such that*

$$\tilde{B}_y = B_y \circ \theta_y.$$

3.2. **Construction of trilinear forms.** Let π_i be an irreducible unitary admissible infinite-dimensional representation of $\mathrm{GL}_2(F)$ with central character ω_i on which we impose the following condition:

$$\text{(Cent)} \qquad \omega_1\omega_2\omega_3 = \epsilon_K.$$

Recall that $\Theta_K^{D_y}(\pi_i)|_{\mathrm{GL}_2(K)} \simeq \pi_{i,K}$. We associate to $y \in \mathcal{Y}_D$ a functional $B_y \in \mathrm{Hom}_{D_y^\times}(\Theta_K^{D_y}(\pi_i), (\omega_i\epsilon_K)\circ \mathrm{N}_{D_y/F})$ by Proposition 3.3. Fix a right $\mathrm{GL}_2(K)$-invariant measure $\mathrm{d}_y\xi$ on $K^\times D_y^\times\backslash \mathrm{GL}_2(K)$. Define an element of

$$\mathrm{Hom}_{\mathrm{GL}_2(K)}(\Theta_K^{D_y}(\pi_1) \otimes \Theta_K^{D_y}(\pi_2) \otimes \Theta_K^{D_y}(\pi_3), \mathbb{C})$$

by the integral

$$(3.1) \qquad \int_{K^\times D_y^\times\backslash \mathrm{GL}_2(K)} B_y(\pi_{1,K}(\xi)\phi_1)B_y(\pi_{2,K}(\xi)\phi_2)B_y(\pi_{3,K}(\xi)\phi_3)\,\mathrm{d}_y\xi$$

for $\phi_i \in \pi_{i,K}$. We will prove the convergence under the following condition:

$$(\bigstar)\qquad \lambda(\pi_1)+\lambda(\pi_2)+\lambda(\pi_3)<\frac{1}{2}.$$

Remark 3.4. (1) Since $\omega_1\omega_2\omega_3=\epsilon_K$, we have

$$\varepsilon(1/2,\pi_{1,K}\times\pi_{2,K}\times\pi_{3,K})=\varepsilon(1/2,\pi_1\times\pi_2\times\pi_3)\varepsilon(1/2,\pi_1^\vee\times\pi_2^\vee\times\pi_3^\vee)=1.$$

Theorem 1.4 of [Pra90] gives

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(K)}(\pi_{1,K}\otimes\pi_{2,K}\otimes\pi_{3,K},\mathbb{C})=1.$$

(2) If π_1,π_2,π_3 are local components of cuspidal automorphic representations, then ($\bigstar$) is fulfilled by the result [KS02] on the Ramanujan estimate for π_i and hence $L(s,\pi_1\times\pi_2\times\pi_3)$ is holomorphic at $s=\frac{1}{2}$.

3.3. **Convergence.**

Lemma 3.5. *If ($\bigstar$) holds, then the integral (3.1) is absolutely convergent.*

Remark 3.6. When $K=F\times F$, the convergence is proved in Lemma 2.1 of [Ich08].

Lemma 3.7 below is stronger than Lemma 3.5. Put

$$E=F\times F\times F,\quad \Pi=\pi_1\otimes\pi_2\otimes\pi_3,\quad \mathbf{U}=\{\mathbf{n}(b)\in\mathrm{GL}_2(E)\mid b\in E\}.$$

Define $\mathrm{T}_{E/F}:E\to F$ by $\mathrm{T}_{E/F}(x,y,z)=x+y+z$ and algebraic groups $U^0\subset\mathbf{G}$ by

$$\mathbf{G}=\{\mathbf{g}\in\mathrm{R}_{E/F}\mathrm{GL}_2\mid\ \det\mathbf{g}\in\mathbb{G}_m\},$$
$$U^0=\{\mathbf{n}(x)\mid x\in\mathrm{R}_{E/F}\mathbb{G}_a,\ \mathrm{T}_{E/F}(x)=0\}.$$

We embed $\mathbf{G}$ diagonally in GSp_6 via the map

$$\iota\left(\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix},\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix},\begin{pmatrix}a_3&b_3\\c_3&d_3\end{pmatrix}\right)=\left(\begin{array}{ccc|ccc}a_1&&&b_1&&\\&a_2&&&b_2&\\&&a_3&&&b_3\\\hline c_1&&&d_1&&\\&c_2&&&d_2&\\&&c_3&&&d_3\end{array}\right).$$

Once and for all we fix a Haar measure $\mathrm{d}g$ on $\mathrm{SL}_2(F)$ and use it to define a Haar measure $\mathrm{d}\mathbf{g}'$ on $\mathrm{SL}_2(E)$. Let $\mathrm{d}z$ and $\mathrm{d}\nu$ be the self-dual Haar measures of F with respect to ψ. We use them to define Haar measures $\mathrm{d}u^0$ on U^0 and $\mathrm{d}\mathbf{u}$ on $\mathbf{U}$. We denote by $\mathrm{d}\ddot{\mathbf{g}}$ and $\mathrm{d}\dddot{\mathbf{g}}$ the quotient measures of $\mathrm{d}\mathbf{g}'$ by $\mathrm{d}u^0$ and $\mathrm{d}\mathbf{u}$, respectively.

Put $\alpha=(y,y)_D$ and $\epsilon=\epsilon_K(\alpha)$. Let

$$W^{\bar\psi}(\Pi)=W^{\bar\psi}(\pi_1)\otimes W^{\bar\psi}(\pi_2)\otimes W^{\bar\psi}(\pi_3)$$

be the Whittaker model of Π with respect to $\bar\psi\circ\mathrm{T}_{E/F}$. Given

$$\mathcal{W}=W_1\otimes W_2\otimes W_3\in W^{\bar\psi}(\Pi),\qquad \Phi=\varphi_1\otimes\varphi_2\otimes\varphi_3\in\mathcal{S}(V_D^3),$$

we consider the integral

$$\mathscr{I}_D^\epsilon(\mathcal{W},\Phi)=\int_{K^\times D_y^\times\backslash\mathrm{GL}_2(K)}\tilde{B}_y(\xi;W_1,\varphi_1)\tilde{B}_y(\xi;W_2,\varphi_2)\tilde{B}_y(\xi;W_3,\varphi_3)\,\mathrm{d}_y\xi.$$

Lemma 3.2 gives the expression

$$\int_{\mathrm{GL}_2(K)/K^\times D_y^\times}\int_{\mathbf{U}\backslash\mathrm{SL}_2(E)}\mathcal{W}^\alpha(\mathbf{d}(\boldsymbol{\nu}(\xi))^{-1}\ddot{\mathbf{g}})\Omega^3_{D,\psi}(\iota(\ddot{\mathbf{g}}))\Phi(\xi y\sigma(\xi)^\iota)\,\mathrm{d}\ddot{\mathbf{g}}\mathrm{d}_y\xi$$

of $\mathscr{I}_D^\epsilon(\mathcal{W},\Phi)$, where $\mathcal{W}^\alpha=W_1^\alpha\otimes W_2^\alpha\otimes W_3^\alpha\in W^{\bar{\psi}^\alpha}(\Pi)$. The measure $\mathrm{d}_y\xi$ is defined in §4.1. If $y'\in\mathcal{Y}_D$ satisfies $\epsilon_K((y',y')_D)=\epsilon$, then Proposition 2.5(1) gives $\xi'\in\mathrm{GL}_2(K)$ such that $y'=\xi'y\sigma(\xi')^\iota$. Since $(y',y')_D=\alpha\boldsymbol{\nu}(\xi')$, it turns out that the integral is independent of the choice of y (cf. Remark 4.1).

Lemma 3.7. *The integral above converges absolutely. Moreover, it defines an element of*

$$\mathrm{Hom}_{\mathrm{GO}(V_D)}(\Theta_K^{D_y}(\pi_1)\otimes\Theta_K^{D_y}(\pi_2)\otimes\Theta_K^{D_y}(\pi_3),\mathbb{C}).$$

Proof. To prove the invariance, it suffices to show that

$$\mathscr{I}_D^\epsilon(\mathcal{W},\Omega^3_{D,\psi}(\rho(a,\mathbf{t}))\Phi)=\mathscr{I}_D^\epsilon(\mathcal{W},\Phi)$$

for $a\in F^\times$ in view of (2.1). Since $\epsilon_K((\rho(a,\mathbf{t})y,\rho(a,\mathbf{t})y)_D)=\epsilon$, it follows from the expression above.

Without loss of generality we may assume that $y=1\in V_D$ in view of Lemma 2.3. Recall the decomposition $V_D=F\oplus\sqrt{\delta}D^\circ$ and

$$\mathrm{GSO}(V_D)=\rho(F^\times\times D_K^\times),\qquad\qquad \mathrm{SO}(\sqrt{\delta}D^\circ)\simeq D^\times/F^\times$$

(see (2.1)). It therefore suffices to show that the integral

$$\int_{\mathrm{SO}(V_D)/\mathrm{SO}(\sqrt{\delta}D^\circ)}\int_{\mathbf{U}\backslash\mathrm{SL}_2(E)}\mathcal{W}(\mathbf{g})\Omega^3_{D,\psi}(\iota(\mathbf{g}))\Phi(h\cdot 1)\,\mathrm{d}\mathbf{g}\mathrm{d}h$$

is absolutely convergent.

Let $\mathfrak{o}_F$ and $\mathfrak{o}_K$ denote the maximal compact subring of F and K, respectively. For simplicity we assume that $2\delta\in\mathfrak{o}_F^\times$. Let $L=V_D\cap\mathrm{M}_2(\mathfrak{o}_K)$ be a maximal integral lattice of V_D. Put

$$C=\{h\in\mathrm{SO}(V_D)\mid hL=L\},\quad L[\mathfrak{a}]=\{x\in V_D\mid(x,x)_D=1,\ (x,L)_D=\mathfrak{a}\}$$

for each fractional ideal $\mathfrak{a}$ of $\mathfrak{o}_F$. Note that $L[\mathfrak{a}]=\emptyset$ unless $\mathfrak{a}\supset\mathfrak{o}_F$ as L is maximal. Fix a generator ϖ of the maximal ideal $\mathfrak{p}$ of $\mathfrak{o}_F$. For each non-negative integer j we choose elements $x_j\in L[\mathfrak{p}^{-j}]$ and $h_j\in\mathrm{SO}(V_D)$ such that $x_j=h_j\cdot 1$. Then $L[\mathfrak{p}^{-j}]=C\cdot x_j$ by Theorem 10.5 of [Shi04]. This combined with Witt's theorem gives the relative Cartan decomposition

$$\mathrm{SO}(V_D)=\bigsqcup_{j=0}^{\infty}C\cdot h_j\mathrm{SO}(\sqrt{\delta}D^\circ).\tag{3.2}$$

Let

$$x_j = \begin{pmatrix} 0 & -\sqrt{\delta}\varpi^j \\ \frac{1}{\sqrt{\delta}\varpi^j} & 0 \end{pmatrix}, \qquad h_j = \rho\left(\varpi^{-j}, \begin{pmatrix} \varpi^j & 0 \\ 0 & 1 \end{pmatrix}\right) h_0.$$

It is enough to prove that the integral

$$\sum_{j=0}^{\infty} q^{2j} \int_C \int_{\mathbf{U}\backslash \mathrm{SL}_2(E)} \mathcal{W}(\mathbf{g}) \Omega^3_{D,\psi}(\iota(\mathbf{g}))\Phi(c\cdot x_j)\,\mathrm{d}\mathbf{g}\mathrm{d}c$$

is absolutely convergent in view of Proposition 2.6 of [KT10]. Equivalently, we will show that the triple integral

$$\sum_{j=0}^{\infty} q^{2j} \int_C \int_{E^\times} \int_{\mathrm{SL}_2(\mathfrak{o}_E)} \mathcal{W}(\mathbf{m}(\mathbf{a})\mathbf{k}) \Omega^3_{D,\psi}(\iota(\mathbf{m}(\mathbf{a})\mathbf{k}))\Phi(c\cdot x_j)|\mathbf{a}|^{-2}\,\mathrm{d}\mathbf{k}\mathrm{d}^\times\mathbf{a}\mathrm{d}c$$

converges absolutely, where $|\mathbf{a}| = |a_1a_2a_3|$ for $\mathbf{a} = (a_1, a_2, a_3) \in E^\times$ and $\mathfrak{o}_E = \mathfrak{o}_F \times \mathfrak{o}_F \times \mathfrak{o}_F$. There exists $\Phi_0 \in \mathcal{S}(E^2)$ and for $\epsilon > 0$ there exists $\Phi_\epsilon \in \mathcal{S}(E)$ such that

$$|\Omega^3_{D,\psi}(\iota(\mathbf{m}(\mathbf{a})\mathbf{k}))\Phi(c\cdot x_j)| \le |\mathbf{a}|^2\Phi_0(\varpi^j\mathbf{a}, \varpi^{-j}\mathbf{a}),$$
$$|\mathcal{W}(\mathbf{m}(\mathbf{a})\mathbf{k})| \le |\mathbf{a}|^{1-\epsilon}|a_1|^{-2\lambda(\pi_1)}|a_2|^{-2\lambda(\pi_2)}|a_3|^{-2\lambda(\pi_3)}\Phi_\epsilon(\mathbf{a}^2)$$

for $\mathbf{a} \in E^\times$, $\mathbf{k} \in \mathrm{SL}_2(\mathfrak{o}_E)$ and $c \in C$. We take ϵ so that

$$1 - 3\epsilon - 2\lambda(\pi_1) - 2\lambda(\pi_2) - 2\lambda(\pi_3) > 0.$$

Then the double integral

$$\int_{F^\times}\int_{E^\times} \frac{|\mathbf{a}|^{1-\epsilon}\Phi_0(t\mathbf{a}, t^{-1}\mathbf{a})\varphi(t)}{|a_1|^{2\lambda(\pi_1)}|a_2|^{2\lambda(\pi_2)}|a_3|^{2\lambda(\pi_3)}}\,|t|^{-2}\mathrm{d}^\times\mathbf{a}\mathrm{d}^\times t$$
$$= \int_{F^\times}\int_{E^\times} \frac{|\mathbf{a}|^{1-\epsilon}\Phi_0(t^2\mathbf{a}, \mathbf{a})\varphi(t)}{|ta_1|^{2\lambda(\pi_1)}|ta_2|^{2\lambda(\pi_2)}|ta_3|^{2\lambda(\pi_3)}}\,|t|^{1-3\epsilon}\mathrm{d}^\times\mathbf{a}\mathrm{d}^\times t$$

converges absolutely for $\varphi \in \mathcal{S}(F)$. We have thus completed our proof. □

4. The trilinear forms and local zeta integrals

4.1. **Garrett's integral representation.** Recall that Z_3 denotes the center of GSp_6 and $E = F \times F \times F$. Put

$$(4.1)\quad \eta = \left(\begin{array}{ccc|ccc} 0&0&0&-1&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ \hline 1&1&1&0&0&0\\ 0&0&0&-1&1&0\\ 0&0&0&-1&0&1 \end{array}\right), \quad w_0 = \left(\begin{array}{ccc|ccc} 0&0&0&0&0&-1\\ 0&0&0&0&-1&0\\ 0&0&0&-1&0&0\\ \hline 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{array}\right).$$

We take Haar measures $\mathrm{d}^\times z = \zeta_F(1)\frac{\mathrm{d}z}{|z|}$ and $\mathrm{d}^\times \nu = \zeta_F(1)\frac{\mathrm{d}\nu}{|\nu|}$ of $F^\times$. Let $\mathrm{d}^\times \dot\nu$ be the Haar measure on $F^\times/F^{\times 2}$ so that

$$\int_{F^\times} f(\nu)\,\mathrm{d}^\times\nu = \int_{F^\times/F^{\times 2}}\int_{F^\times} f(z^2\dot\nu)\,\mathrm{d}z\mathrm{d}^\times\dot\nu$$

for $f \in L^1(F^\times)$. We define a Haar measure $\mathrm{d}\mathbf{g}$ on $Z_3U^0\backslash\mathbf{G}$ by

$$\int_{Z_3U^0\backslash\mathbf{G}} f(\mathbf{g})\,\mathrm{d}\mathbf{g} = \int_{F^\times/F^{\times 2}}\int_{U^0\backslash\mathrm{SL}_2(E)} |\dot\nu|^2 f(\mathbf{d}(\dot\nu)\ddot{\mathbf{g}})\,\mathrm{d}\ddot{\mathbf{g}}\mathrm{d}\dot\nu \tag{4.2}$$

for $f \in L^1(Z_3U^0\backslash\mathbf{G})$. Let $\mathrm{d}x$ be the self-dual Haar measure on V_D with respect to $\psi((x,y)_D)$. We take the measure $\mathrm{d}^\times x = \zeta_F(1)L(2,\epsilon_K)|(x,x)_D|^{-2}\mathrm{d}x$ on $\mathcal{Y}_D$, which is invariant under the action ρ_1 of $D_K^\times$. For each $y \in \mathcal{Y}_D$ we define a map $\wp_y : D_K^\times \to \mathcal{Y}_D$ by $\wp_y(\xi) = \rho_1(\xi)y = \xi y\sigma(\xi)^\iota$ for $\xi \in D_K^\times$. Put

$$\mathcal{U}_y = \{\xi \in D_K^\times \mid \wp_y(\xi) = y\}.$$

We obtain a $D_K^\times$-invariant measure $\mathrm{d}_y\xi'$ on $D_K^\times/\mathcal{U}_y$ by the pull-back $\mathrm{d}_y\xi' = \wp_y^*\mathrm{d}^\times x$. In view of Proposition 2.5(3) we obtain a measure $\mathrm{d}_y\xi$ on $D_K^\times/K^\times D_y^\times$ as the quotient of $\mathrm{d}_y\xi'$ by $\mathrm{d}^\times z$.

Remark 4.1. Given $h \in D_K^\times$, we define $\iota_h : D_K^\times \to D_K^\times$ by $\iota_h(\xi) = h\xi h^{-1}$ for $\xi \in D_K^\times$. Since $\iota_h^* \circ \wp_{\rho_1(h)y}^* = \wp_y^* \circ \rho_1(h)^*$, we have

$$\iota_h^*\mathrm{d}_{\rho_1(h)y}\xi = \mathrm{d}_y\xi.$$

When $\epsilon = \epsilon_K((y,y)_D)$, we have

$$\int_{\mathcal{Y}_D^\epsilon} f(x)\,\mathrm{d}x = \int_{D_K^\times/K^\times D_y^\times}\int_{F^\times} f(z\wp_y(\xi))\frac{|z^2\boldsymbol{\nu}(\xi)(y,y)_D|^2}{\zeta_F(1)L(2,\epsilon_K)}\mathrm{d}^\times z\mathrm{d}_y\xi \tag{4.3}$$

for any Schwartz function f on V_D.

Let π_1,π_2,π_3 be irreducible unitary generic admissible representations of $\mathrm{GL}_2(F)$ which satisfy (Cent) and ($\bigstar$). Put $\mathbf{G}^\star = \mathbf{G} \cap \mathrm{GSp}_6(F)^\star$ (see Definition 2.7). Given $\alpha \in F^\times$ and a Whittaker function $\mathcal{W} \in W^{\bar\psi}(\Pi)$ with respect to $\bar\psi \circ \mathrm{T}_E$, we define $\mathcal{W}^\alpha \in W^{\bar\psi^\alpha}(\Pi)$ by

$$\mathcal{W}^\alpha(\mathbf{g}) = \mathcal{W}(\mathbf{d}(\alpha)^{-1}\mathbf{g}).$$

Let $\mathcal{K}$ be the standard maximal compact subgroup of $\mathrm{GSp}_6(F)$. Let $I_3(s,\epsilon_K)$ be the normalized induced representation of $\mathrm{GSp}_6(F)$, consisting of all smooth right $\mathcal{K}$-finite functions $f^{(s)} : \mathrm{GSp}_6(F) \to \mathbb{C}$ such that

$$f^{(s)}(\mathbf{d}(t)\mathbf{n}(z)\mathbf{m}(a)g) = \epsilon_K(\det a)|t^{-3}(\det a)^2|^{s+1}f^{(s)}(g).$$

We associate to $\Phi \in \mathcal{S}(V_D^3)$ a function f_Φ on $\mathrm{GSp}_6(F)^\star$ defined by

$$f_\Phi(g) = \Omega^3_{D,\psi}(h,g)\Phi(0),$$

where $h \in \mathrm{GO}(V_D)$ is chosen so that $\boldsymbol{\nu}(h) = \nu_3(g)$. The right hand side is independent of the choice of h. Since it satisfies

$$f_\Phi(\mathbf{d}(t)\mathbf{n}(b)\mathbf{m}(a)g) = \epsilon_K(\det a)|t^{-3}(\det a)^2|f_\Phi(g)$$

for $a \in \mathrm{GL}_3(F)$, $b \in \mathrm{Sym}_3(F)$ and $t \in \mathrm{N}_{K/F}(K^\times)$, it is uniquely extended to an element $f_\Phi^{(0)}$ of $I_3(0, \epsilon_K)$.

The local integral

$$Z(\mathcal{W}, f_\Phi^{(0)}) = \int_{Z_3U^0\backslash \mathbf{G}} \mathcal{W}(\mathbf{g}) f_\Phi^{(0)}(\eta\iota(\mathbf{g}))\,\mathrm{d}\mathbf{g}$$

converges absolutely by ($\bigstar$) and [Ike92, Lemma 2.1]. If ψ is of order zero, ϵ_K is unramified, Φ is the characteristic function of $V_D \cap \mathrm{M}_2(\mathfrak{o}_K)$, $\mathcal{W}(\mathrm{GL}_2(\mathfrak{o}_E)) = 1$ and $\mathrm{vol}(\mathfrak{o}_F^\times U^0(\mathfrak{o}_F)\backslash \mathbf{G}(\mathfrak{o}_F), \mathrm{d}\mathbf{g}) = 1$, then by Theorem 3.1 of [PSR87]

$$Z(\mathcal{W}, f_\Phi^{(0)}) = \frac{L\left(\frac{1}{2}, \Pi\right)}{\zeta_F(2) L(2, \epsilon_K)}. \tag{4.4}$$

4.2. **Partial zeta integrals and trilinear forms.** If $K \simeq F \times F$, then $\gamma(\Pi) = \varepsilon(\Pi)$ is a sign, and hence $\gamma(\Pi) \neq -\epsilon(D)$ if and only if $\varepsilon(\Pi) = \epsilon(D)$ if and only if $\mathscr{I}_D^+$ is non-vanishing by epsilon dichotomy. Therefore we will assume that K is a quadratic extension of a non-archimedean local field F. Let $\alpha \in F^\times$. We introduce the partial zeta integral

$$Z^\star(\mathcal{W}^\alpha, \Phi) = \int_{Z_3U^0\backslash \mathbf{G}^\star} \mathcal{W}^\alpha(\mathbf{g}) f_\Phi(\eta\iota(\mathbf{g}))\,\mathrm{d}\mathbf{g}.$$

Observe that

$$f_\Phi^{(0)}(\eta\iota(\mathbf{d}(\alpha)g_1, \mathbf{d}(\alpha)g_2, \mathbf{d}(\alpha)g_3)) = \epsilon_K(\alpha)|\alpha|^{-1} f_\Phi^{(0)}(\eta\iota(g_1, g_2, g_3))$$

and hence by the change of variables $\mathbf{g} \mapsto \mathbf{d}(\alpha)\mathbf{g}\mathbf{d}(\alpha)^{-1}$

$$\begin{aligned} Z^\star(\mathcal{W}^\alpha, \Phi) &= |\alpha|^2 \int_{Z_3U^0\backslash \mathbf{G}^\star} \mathcal{W}(\mathbf{g}\mathbf{d}(\alpha)^{-1}) f_\Phi^{(0)}(\eta\iota(\mathbf{d}(\alpha)\mathbf{g}\mathbf{d}(\alpha)^{-1}))\,\mathrm{d}\mathbf{g} \\ &= \epsilon_K(\alpha)|\alpha| \int_{Z_3U^0\backslash \mathbf{G}^\star} \mathcal{W}(\mathbf{g}\mathbf{d}(\alpha)^{-1}) f_\Phi^{(0)}(\eta\iota(\mathbf{g}\mathbf{d}(\alpha)^{-1}))\,\mathrm{d}\mathbf{g}. \end{aligned} \tag{4.5}$$

When $\epsilon_K(\alpha) = -1$, we get

$$Z(\mathcal{W}, f_\Phi^{(0)}) = Z^\star(\mathcal{W}, \Phi) - |\alpha|^{-1} Z^\star(\mathcal{W}^\alpha, \Phi).$$

Proposition 4.2. *Let $\alpha \in F^\times$, $\Phi \in \mathcal{S}(V_D^3)$ and $\mathcal{W} \in W^{\bar{\psi}}(\Pi)$. Then*

$$Z^\star(\mathcal{W}^\alpha, \Phi) = |\alpha| \epsilon(D) L(2, \epsilon_K)^{-1} \mathscr{I}_D^{\epsilon_K(\alpha)}(\mathcal{W}, \Phi).$$

Corollary 4.3. Let $\Phi \in \mathcal{S}(V_D^3)$ and $\mathcal{W} \in W^{\bar{\psi}}(\Pi)$. Then

$$Z(\mathcal{W}, f_\Phi^{(0)}) = \epsilon(D) L(2, \epsilon_K)^{-1}(\mathscr{I}_D^+(\mathcal{W}, \Phi) - \mathscr{I}_D^-(\mathcal{W}, \Phi)).$$

We will prove Proposition 4.2 in §4.5. Granted Proposition 4.2, we can easily prove Theorems 1.1 and 1.2.

4.3. **The proof of Theorem 1.1.** Let π_1, π_2, π_3 be irreducible unitary generic admissible representations of $\mathrm{GL}_2(F)$ which satisfy (Cent) and ($\bigstar$). Define the intertwining operator $M(s) : I_3(s, \epsilon_K) \to I_3(-s, \epsilon_K) \otimes \epsilon_K \circ \nu_3$ by

$$M(s)f^{(s)}(g) = |2|^{-3/2} \int_{\mathrm{Sym}_3(F)} f^{(s)}(w_0^{-1}\mathbf{n}(b)g)\,\mathrm{d}b,$$

where w_0 is defined in (4.1) and $\mathrm{d}b$ is the self-dual Haar measure of $\mathrm{Sym}_3(F)$ with respect to $(b, b') \mapsto \psi(\mathrm{tr}(bb'))$. This integral is absolutely convergent for $\Re(s) \gg 0$ and can be meromorphically continued to the whole complex plane. We normalize the operator $M(s)$ by setting

$$M^*(s) = \gamma(2s - 1, \epsilon_K, \psi)\gamma(4s - 1, 1, \psi)M(s).$$

The gamma factor $\gamma(s, \Pi, \psi)$ is defined as the proportionality constant of the functional equation

$$Z(\mathcal{W}, M^*(s)f^{(s)}) = \gamma\left(s + \frac{1}{2}, \Pi, \psi\right) Z(\mathcal{W}, f^{(s)})$$

for $f^{(s)} \in I_3(s, \epsilon_K)$. This gamma factor coincides with $\gamma(s, \sigma_1 \otimes \sigma_2 \otimes \sigma_3, \psi)$ by Proposition 3.3.7 of [Ram00] (cf. [CCI20]), where σ_i be the 2-dimensional representation of the Weil-Deligne group of F associated to π_i by the local Langlands correspondence for GL_2. The central value $\gamma(\Pi) = \gamma\big(\frac{1}{2}, \Pi, \psi\big)$ is independent of the choice of ψ (see Remark 1.4(2)).

Theorem 4.4. *The following conditions are equivalent:*

- $\mathscr{I}_D^\epsilon \in \mathrm{Hom}_{\mathrm{GL}_2(K)}(\pi_{1,K} \otimes \pi_{2,K} \otimes \pi_{3,K}, \mathbb{C})$ *is zero;*
- $\gamma(\Pi) = -\epsilon \cdot \epsilon(D)$.

Theorem 4.4 can be deduced from Lemma 4.5 and Proposition 4.7 below.

Lemma 4.5. *There are* $\mathcal{W} \in W^{\bar{\psi}}(\Pi)$ *and* $\Phi \in \mathcal{S}(V_D^3)$ *such that not both* $\mathscr{I}_D^+(\mathcal{W}, \Phi)$ *and* $\mathscr{I}_D^-(\mathcal{W}, \Phi)$ *are zero.*

Proof. Given $\alpha \in F^\times$, we define an element $f_\Phi^\alpha \in I_3(0, \epsilon_K)$ by $f_\Phi^\alpha(g) = f_\Phi^{(0)}(g\mathbf{d}(\alpha))$ for $g \in \mathrm{GSp}_6(F)$. Put $R_3(V_D) := \{f_\Phi \mid \Phi \in \mathcal{S}(V_D^3)\}$. Fix $\alpha_0 \in F^\times$ with $\epsilon_K(\alpha_0) = -1$. Theorem 2.1 of [KR94] tells us that

$$I_3(0, \epsilon_K) = R_3(V_D) \oplus R_3(V_D^{\alpha_0}).$$

Since $R_3(V_D^{\alpha_0}) = \{f_\Phi^{\alpha_0} \mid \Phi \in \mathcal{S}(V_D^3)\}$, the space $I_3(0, \epsilon_K)$ is a $\mathbb{C}$-linear span of elements of the form f_Φ^α.

If $\mathscr{I}_D^+(\mathcal{W}, \Phi) = \mathscr{I}_D^-(\mathcal{W}, \Phi) = 0$ for all $\mathcal{W} \in W^{\bar{\psi}}(\Pi)$ and $\Phi \in \mathcal{S}(V_D^3)$, then $Z^\star(\mathcal{W}^\alpha, \Phi) = 0$ for all $\alpha \in F^\times$ by Proposition 4.2, and hence $Z(\mathcal{W}, f_\Phi^\alpha) = 0$ for all $\alpha \in F^\times$ by (4.5). This is a contradiction as the zeta integral defines a non-zero functional on $\Pi \otimes I_3(0, \epsilon_K)$ by Proposition 3.3 of [PSR87]. $\square$

Let $S \in \mathrm{Sym}_3(F)$ with $\det S \neq 0$. For a section $f^{(s)}$ of $I_3(s, \epsilon_K)$ we put

$$W_S(f^{(s)}) = \int_{\mathrm{Sym}_3(F)} f^{(s)}(w_0\mathbf{n}(b))\psi(-\mathrm{tr}(Sb))\,\mathrm{d}b.$$

The integral can be continued to an entire function in s. Let $\epsilon(S)$ be either 1 or -1 according to whether S is split or anisotropic. Theorem 2.1 combined with Lemma 3.1 of [Ike17] gives

$$W_S(M^*(s)f^{(s)}) = \epsilon(S)\epsilon_K(4\det S)|4\det S|^{-2s}W_S(f^{(s)}).$$

Lemma 4.6. *Let $\Phi \in \mathcal{S}(V_D^3)$. Then*

$$M^*(0)f_\Phi^{(0)} = \epsilon_K(-1)\epsilon(D)f_\Phi^{(0)} \cdot \epsilon_K \circ \nu_3.$$

Proof. It suffices to determine $M^*(0)f_\Phi^{(0)}|_{\mathrm{Sp}_6(F)}$. The operator $M^*(0)$ preserves the space $R_3(V_D)$ by Proposition 5.5 of [KR92]. Since this space is irreducible as an $\mathrm{Sp}_6(F)$-module by Corollary 3.7 of [KR92], the operator $M^*(0)$ acts on it by scalar multiplication. Take S such that $\epsilon(S) = 1$ and $\epsilon_K(-4\det S) = \epsilon(D)$. Then $W_S(M^*(0)f_\Phi) = \epsilon_K(-1)\epsilon(D)W_S(f_\Phi)$. Since such an S is represented by V_D (cf. Remark 2.4), Proposition 2.7 of [KR94] gives $\Phi \in \mathcal{S}(V_D^3)$ with $W_S(f_\Phi) \neq 0$. □

Proposition 4.7. *For all $\mathcal{W} \in W^{\bar\psi}(\Pi)$ and $\Phi \in \mathcal{S}(V_D^3)$ we have*

$$(1 - \epsilon_K(-1)\epsilon(D)\gamma(\Pi))\mathscr{I}_D^+(\mathcal{W},\Phi) = -(1 + \epsilon_K(-1)\epsilon(D)\gamma(\Pi))\mathscr{I}_D^-(\mathcal{W},\Phi).$$

Proof. Take $\alpha \in F^\times$ with $\epsilon_K(\alpha) = -1$. Then we have

$$\begin{aligned} Z(\mathcal{W}, M^*(0)f_\Phi^{(0)}) &= \epsilon_K(-1)\epsilon(D)Z(\mathcal{W}, f_\Phi^{(0)} \cdot \epsilon_K \circ \nu_3) \\ &= \epsilon_K(-1)\epsilon(D)(Z^\star(\mathcal{W},\Phi) + |\alpha|^{-1}Z^\star(\mathcal{W}^\alpha,\Phi)) \\ &= \epsilon_K(-1)L(2,\epsilon_K)^{-1}(\mathscr{I}_D^+(\mathcal{W},\Phi) + \mathscr{I}_D^-(\mathcal{W},\Phi)) \end{aligned}$$

by Lemma 4.6, (4.5) and Proposition 4.2.

We combine Corollary 4.3 with the functional equation

$$Z(\mathcal{W}, M^*(0)f_\Phi^{(0)}) = \gamma(\Pi)Z(\mathcal{W}, f_\Phi^{(0)})$$

to verify the relation. □

Corollary 4.3 and Proposition 4.7 give the following result:

Corollary 4.8. Let $\Phi \in \mathcal{S}(V_D^3)$ and $\mathcal{W} \in W^{\bar\psi}(\Pi)$. If $\gamma(\Pi) \neq -\epsilon_K(-1)\epsilon(D)$, then

$$Z(\mathcal{W}, f_\Phi^{(0)}) = \frac{2}{\epsilon(D) + \epsilon_K(-1)\gamma(\Pi)} \cdot L(2,\epsilon_K)^{-1}\mathscr{I}_D^+(\mathcal{W},\Phi).$$

4.4. **The proof of Theorem 1.2.** Letting $y = 1 \in V_D$, we put

$$B_i^\natural = \zeta_F(2)L(1, \mathrm{Ad}(\pi_i) \otimes \epsilon_K)^{-1}B_y \in \mathrm{Hom}_{D^\times}(\pi_{i,K}^D, (\omega_i\epsilon_K) \circ \mathrm{N}_{D/F}),$$

where B_y is as in Proposition 3.3, and define $\theta : \Pi \otimes \mathcal{S}(V_D^3) \to \Pi_K^D$ by

$$\theta(\mathcal{W},\Phi) = \theta_y(W_1,\varphi_1) \otimes \theta_y(W_2,\varphi_2) \otimes \theta_y(W_3,\varphi_3)$$

for $\mathcal{W} = W_1 \otimes W_2 \otimes W_3$ and $\Phi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$. We define an element

$$\mathbf{I}^\natural \in \mathrm{Hom}_{D_K^\times}(\pi_{1,K}^D \otimes \pi_{2,K}^D \otimes \pi_{3,K}^D, \mathbb{C})$$

by the convergent integral

$$\mathbf{I}^\natural(\phi_1\otimes\phi_2\otimes\phi_3)=\int_{K^\times D^\times\backslash D_K^\times}B_1^\natural(\pi_{1,K}^D(\xi)\phi_1)B_2^\natural(\pi_{2,K}^D(\xi)\phi_2)B_3^\natural(\pi_{3,K}^D(\xi)\phi_3)\,\mathrm{d}\xi,$$

where $\mathrm{d}\xi=\mathrm{d}_1\xi$. Assuming that $\gamma(\Pi)\neq-\epsilon_K(-1)\epsilon(D)$, we normalize $\mathbf{I}^\natural$ by

$$I^\natural=\frac{L(1,\mathrm{Ad}(\Pi)\otimes\epsilon_K)}{\zeta_F(2)^2L\left(\frac{1}{2},\Pi\right)}\cdot\frac{2}{\epsilon(D)+\epsilon_K(-1)\gamma(\Pi)}\cdot\mathbf{I}^\natural.$$

By the definition of $B_i^\natural$ the functionals $\mathbf{I}^\natural$ and $\mathscr{I}_D^+$ are related as follows:

$$\mathscr{I}_D^+(\mathcal{W},\Phi)=\frac{L(1,\mathrm{Ad}(\Pi)\otimes\epsilon_K)}{\zeta_F(2)^3}\mathbf{I}^\natural(\theta(\mathcal{W},\Phi))=\frac{L\left(\frac{1}{2},\Pi\right)}{\zeta_F(2)}\cdot\frac{\epsilon(D)+\epsilon_K(-1)\gamma(\Pi)}{2}I^\natural.$$

Corollary 4.8 gives

$$Z(\mathcal{W},f_\Phi^{(0)})=\frac{L\left(\frac{1}{2},\Pi\right)}{\zeta_F(2)L(2,\epsilon_K)}I^\natural(\theta(\mathcal{W},\Phi)). \tag{4.6}$$

Now we assume that $\epsilon_K,\pi_1,\pi_2,\pi_3$ are unramified, ψ has order 0 and

$$\epsilon(D)=1,\qquad \mathrm{vol}(\mathfrak{o}_F^\times U^0(\mathfrak{o}_F)\backslash\mathbf{G}(\mathfrak{o}_F),\mathrm{d}\ddot{\mathbf{g}})=1,$$
$$W_i(\mathrm{GL}_2(\mathfrak{o}_F))=1,\qquad \mathrm{vol}(\mathfrak{o}_K^\times\mathrm{GL}_2(\mathfrak{o}_F)\backslash\mathrm{GL}_2(\mathfrak{o}_K),\mathrm{d}\xi)=1.$$

Let φ_i be the characteristic function of $V^+\cap\mathrm{M}_2(\mathfrak{o}_K)$. Then

$$B_i^\natural(\theta_y(W_i,\varphi_i))=1,\qquad I^\natural(\theta(\mathcal{W},\Phi))=1$$

by Proposition 2.11(1) and (4.4).

4.5. **The proof of Proposition 4.2.** The rest of this section is devoted to the proof of Proposition 4.2. The proof is similar to that of Proposition 5.1 of [Ich08] but more complicated as Proposition 2.5(1) says that the action of $\mathrm{GL}_2(K)$ divides $\mathcal{Y}_D$ into two orbits $\mathcal{Y}_D=\mathcal{Y}_D^+\sqcup\mathcal{Y}_D^-$, where

$$\mathcal{Y}_D^\pm=\{y\in\mathcal{Y}_D\mid\epsilon_K((y,y)_D)=\pm1\}.$$

Recall that $\boldsymbol{\nu}(\xi)=\mathrm{N}_{K/F}(\det\xi)$ for $\xi\in\mathrm{GL}_2(K)$. We associate to $\Phi\in\mathcal{S}(V_D^3)$ a function $H_\psi^D(\Phi):\mathrm{GL}_2(K)\times\mathrm{SL}_2(E)\to\mathbb{C}$ by

$$H_\psi^D(\xi,g;\Phi)=L(2,\epsilon_K)^{-1}(\Omega_{D,\psi}^3(\iota(g))\Phi)(\xi\sigma(\xi)^\iota).$$

Take $y\in\mathcal{Y}_D$. Define $\Phi_y\in\mathcal{S}(V_{D_y}^3)$ by $\Phi_y(x)=\Phi(xy)$ for $x\in V_{D_y}^3$. Put

$$\gamma=(y,y)_D,\quad \epsilon=\epsilon_K(\gamma),\quad \boldsymbol{\nu}_3=\nu_3\circ\iota,$$
$$\begin{aligned}J_\Phi^\epsilon(g)&=J_\Phi^y(g)\\&=|\gamma|^2\int_{\mathrm{GL}_2(K)/K^\times D_y^\times}\int_{F^\times}H_{\psi^\gamma}^{D_y}(\xi,\mathbf{m}(z)g;\Phi_y)\epsilon_K(z)\left|\frac{\boldsymbol{\nu}(\xi)}{z}\right|^2\frac{\mathrm{d}^\times z}{\zeta_F(1)}\mathrm{d}_y\xi\end{aligned}$$

for $g\in\mathrm{SL}_2(E)$. The integral makes sense by Proposition 2.5(3).

Lemma 4.9. *Let $\Phi\in\mathcal{S}(V_D^3)$. Then for $\mathbf{g}\in\mathbf{G}^\star$*

$$f_\Phi^{(0)}(\eta\mathbf{g})=\epsilon(D)|\boldsymbol{\nu}_3(\mathbf{g})|^{-1}(J_\Phi^+(\mathbf{d}(\boldsymbol{\nu}_3(\mathbf{g})^{-1})\mathbf{g})+J_\Phi^-(\mathbf{d}(\boldsymbol{\nu}_3(\mathbf{g})^{-1})\mathbf{g})).$$

Proof. Since $f^{(0)}_\Phi(\eta\mathbf{d}(\nu)g) = \frac{\epsilon_K(\nu)}{|\nu|}f^{(0)}_\Phi(\eta g)$ for all $\nu \in F^\times$ and $g \in \mathrm{Sp}_6(F)$, we may assume that $\mathbf{g} \in \mathrm{SL}_2(E)$. Put

$$w_1 = \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & \mathbf{1}_2 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_2 \end{array}\right), \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\eta = w_1\mathbf{m}(A)$. Recall that

$$\epsilon(D)\Omega^1_{D,\psi}(J_1)\varphi(y) = \int_{V_D}\varphi(x)\psi((x,y)_D)\,\mathrm{d}x = \int_{\mathcal{Y}_D}\varphi(x)\psi((x,y)_D)\,\mathrm{d}x$$

for $\varphi \in \mathcal{S}(V_D)$. We see therefore that for $g \in \mathrm{Sp}_6(F)$

$$\begin{aligned}\epsilon(D)f^{(0)}_\Phi(\eta g) &= \epsilon(D)\Omega^3_{D,\psi}(w_1\mathbf{m}(A)g)\Phi(0) \\ &= \int_{\mathcal{Y}_D}\Omega^3_{D,\psi}(\mathbf{m}(A)g)\Phi(x,0,0)\,\mathrm{d}x = \int_{\mathcal{Y}_D}\Omega^3_{D,\psi}(g)\Phi(x,x,x)\,\mathrm{d}x.\end{aligned}$$

Take $y \in \mathcal{Y}_D$. Employing (4.3), we rewrite the right hand side as a sum of

$$\int_{\mathrm{GL}_2(K)/K^\times D_y^\times}\int_{F^\times}\Omega^3_{D,\psi}(\iota(g))\Phi(z\wp_y(\xi))\frac{|z^2\boldsymbol{\nu}(\xi)\gamma|^2}{\zeta_F(1)L(2,\epsilon_K)}\,\mathrm{d}^\times z\,\mathrm{d}_y\xi.$$

It is equal to $J^y_\Phi(g)$ by (2.2) and Remark 3.1. □

To simplify notation, we put

$$\mathcal{B} = \mathrm{N}_{K/F}(K^\times), \qquad \mathcal{B}^c = F^\times \setminus \mathcal{B}, \qquad \mathcal{B}_E = \mathcal{B}\times\mathcal{B}\times\mathcal{B}.$$

Let $\mathbb{R}_{\geq 0}$ be the set of non-negative real numbers. Fix a $\mathbb{R}_{\geq 0}$-valued function $\beta \in C^\infty_c(E^\times)$ whose support is contained in $\mathcal{B}_E$ and such that $\beta(au) = \beta(a)$ for $a \in \mathcal{B}_E$ and $u \in \mathfrak{o}_E^\times \cap \mathcal{B}_E$. Let $\mathcal{C} = \mathfrak{o}_F$. We choose a $\mathbb{R}_{\geq 0}$-valued function $\phi \in C^\infty_c(F)$ so that

$$\begin{aligned}&\phi(1) = 0, &&\mathrm{supp}(\phi)\cdot\mathcal{C} \subset \mathrm{supp}(\phi), \\ &\hat{\phi}_\alpha(0) = 1, &&\mathrm{supp}(\phi)\cap(1-\mathcal{B}^c) = \emptyset\end{aligned}$$

and such that $\hat{\phi}_\alpha(x+v) = \hat{\phi}_\alpha(x)$ for $x \in F$ and $v \in \mathrm{T}_{E/F}(\mathrm{supp}(\beta)\cdot\mathfrak{o}_E)$. Here $\hat{\phi}_\alpha \in \mathcal{S}(F)$ is the Fourier transform of ϕ defined by

$$\hat{\phi}_\alpha(b) = \int_F \phi(z)\psi^\alpha(zb)\,\mathrm{d}_\alpha z$$

for $b \in F$, where $\mathrm{d}_\alpha z = |\alpha|^{1/2}\mathrm{d}z$ is the self-dual Haar measure of F with respect to ψ^α. We can define a function $\tau^\alpha_{\beta,\phi}$ on $Z_3U^0\backslash\mathbf{G}^\star$ by

$$\tau^\alpha_{\beta,\phi}\left(\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}k\right) = \hat{\phi}_\alpha(\mathrm{T}_{E/F}(b))\beta(ad^{-1})$$

for $b \in E$, $a, d \in E^\times$ with $ad \in \mathcal{B}$ and $k \in \mathrm{GL}_2(\mathfrak{o}_E)\cap\mathbf{G}^\star$. One can easily verify that $\tau^\alpha_{\beta,\phi}$ is well-defined.

We define a modified truncated partial zeta integral by

$$Z^\star_{\beta,\phi}(\mathcal{W}^\alpha,\Phi)=\int_{Z_3U^0\backslash\mathbf{G}^\star}\mathcal{W}^\alpha(\mathbf{g})f_\Phi(\eta\iota(\mathbf{g}))\tau^\alpha_{\beta,\phi}(\mathbf{g})\,\mathrm{d}\mathbf{g}.$$

Since $Z^\star(\mathcal{W}^\alpha,\Phi)$ is absolutely convergent and $\tau^\alpha_{\beta,\phi}$ is bounded, this integral is absolutely convergent. Put

$$I^{\alpha,\epsilon}_{\beta,\phi}=I^{\alpha,y}_{\beta,\phi}=\int_{Z_3U^0\backslash\mathbf{G}^\star}\mathcal{W}^\alpha(\mathbf{g})\tau^\alpha_{\beta,\phi}(\mathbf{g})|\boldsymbol{\nu}_3(\mathbf{g})|^{-1}J^y_\Phi(\mathbf{d}(\boldsymbol{\nu}_3(\mathbf{g})^{-1})\mathbf{g})\,\mathrm{d}\mathbf{g},$$

where $\epsilon=\epsilon_K((y,y)_D)$. Then Lemma 4.9 gives

$$Z^\star_{\beta,\phi}(\mathcal{W}^\alpha,\Phi)=\epsilon(D)(I^{\alpha,+}_{\beta,\phi}+I^{\alpha,-}_{\beta,\phi}).$$

Following [Ich08], for given $\mathbf{g}\in\mathbf{G}^\star$, we put

$$\mathcal{W}^\alpha_{\beta,\phi}(\mathbf{g})=\int_F\int_{F^\times}\mathcal{W}^\alpha(\mathbf{d}(\nu)\mathbf{g})\tau^\alpha_{\beta,\phi}(\mathbf{n}(z/3)\mathbf{d}(\nu)\mathbf{g})\psi^\alpha(\nu z-z)|\nu|^2\,\frac{\mathrm{d}^\times_\alpha\nu}{\zeta_F(1)}\mathrm{d}_\alpha z.$$

Then $\mathcal{W}^\alpha_{\beta,\phi}(\mathbf{n}(x)\mathbf{g})=\psi^\alpha(-\mathrm{T}_{E/F}(x))\mathcal{W}^\alpha_{\beta,\phi}(\mathbf{g})$ for $x\in E$ and $\mathbf{g}\in\mathbf{G}^\star$.

Lemma 4.10. *(1) If $\epsilon_K(\alpha)\neq\epsilon$, then $I^{\alpha,\epsilon}_{\beta,\phi}=0$.*

(2) If $\alpha=(y,y)_D$, then

$$I^{\alpha,y}_{\beta,\phi}=|\alpha|\int_{\mathrm{GL}_2(K)/K^\times D_y^\times}\int_{\mathbf{U}\backslash\mathrm{SL}_2(E)}H^{D_y}_{\psi^\alpha}(\xi,\ddot{\mathbf{g}};\Phi_y)\mathcal{W}^\alpha_{\beta,\phi}(\mathbf{d}(\boldsymbol{\nu}(\xi)^{-1})\ddot{\mathbf{g}})\,\mathrm{d}\ddot{\mathbf{g}}\,\mathrm{d}_y\xi.$$

Proof. To simplify notation, we write $\mathfrak{X}_y=\mathrm{GL}_2(K)/K^\times D_y^\times$. Changing the variables $\mathbf{g}\mapsto\mathbf{m}(z^{-1})\mathbf{g}$, we get

$$\begin{aligned}I^{\alpha,y}_{\beta,\phi}=&|\gamma|^2\int_{Z_3U^0\backslash\mathbf{G}^\star}\int_{\mathfrak{X}_y}\int_{F^\times}\mathcal{W}^\alpha(\mathbf{m}(z^{-1})\mathbf{g})\tau^\alpha_{\beta,\phi}(\mathbf{m}(z^{-1})\mathbf{g})|\boldsymbol{\nu}_3(\mathbf{g})|^{-1}\\ &\times\epsilon_K(z)H^{D_y}_{\psi^\gamma}(\xi,\mathbf{d}(\boldsymbol{\nu}_3(\mathbf{g})^{-1})\mathbf{g};\Phi_y)|z\boldsymbol{\nu}(\xi)|^2\,\frac{\mathrm{d}^\times z}{\zeta_F(1)}\mathrm{d}_y\xi\mathrm{d}\mathbf{g}\\ =&|\gamma|^2\int_{F^{\times2}\backslash\mathcal{B}}\int_{U^0\backslash\mathrm{SL}_2(E)}\int_{\mathfrak{X}_y}\int_{F^\times}\mathcal{W}^\alpha(\mathbf{d}(z^2\dot\nu)\ddot{\mathbf{g}})\tau^\alpha_{\beta,\phi}(\mathbf{d}(z^2\dot\nu)\ddot{\mathbf{g}})\\ &\times H^{D_y}_{\psi^\gamma}(\xi,\ddot{\mathbf{g}};\Phi_y)|z^2\dot\nu\boldsymbol{\nu}(\xi)^2|\,\frac{\mathrm{d}^\times z}{\zeta_F(1)}\mathrm{d}_y\xi\mathrm{d}\ddot{\mathbf{g}}\mathrm{d}^\times\dot\nu\end{aligned}$$

by (4.2). Combining the integrals over $F^{\times2}\backslash\mathcal{B}$ and $F^\times$ into an integral over $\mathcal{B}$ and integrating over $U^0\backslash\mathbf{U}$, we obtain

$$\begin{aligned}\frac{I^{\alpha,y}_{\beta,\phi}}{|\gamma|^2}=&\int_{U^0\backslash\mathrm{SL}_2(E)}\int_{\mathfrak{X}_y}\int_{\mathcal{B}}\mathcal{W}^\alpha(\mathbf{d}(\nu)\ddot{\mathbf{g}})\tau^\alpha_{\beta,\phi}(\mathbf{d}(\nu)\ddot{\mathbf{g}})H^{D_y}_{\psi^\gamma}(\xi,\ddot{\mathbf{g}};\Phi_y)|\nu\boldsymbol{\nu}(\xi)^2|\frac{\mathrm{d}^\times\nu}{\zeta_F(1)}\mathrm{d}_y\xi\mathrm{d}\ddot{\mathbf{g}}\\ =&\int_{\mathfrak{X}_y}\int_{\mathbf{U}\backslash\mathrm{SL}_2(E)}H^{D_y}_{\psi^\gamma}(\xi,\ddot{\mathbf{g}};\Phi_y)|\boldsymbol{\nu}(\xi)|^2\int_{\mathcal{B}}\mathcal{W}^\alpha(\mathbf{d}(\nu)\ddot{\mathbf{g}})L^{\alpha,y}_{\beta,\phi}(\ddot{\mathbf{g}},\xi,\nu)\,\frac{\mathrm{d}^\times_\alpha\nu}{\zeta_F(1)}\mathrm{d}\ddot{\mathbf{g}}\mathrm{d}_y\xi,\end{aligned}$$

where

$$L^{\alpha,y}_{\beta,\phi}(\mathbf{g},\xi,\nu)=|\alpha^{-1}\nu|\int_F\overline{\psi^\alpha(z/\nu)}\tau^\alpha_{\beta,\phi}(\mathbf{d}(\nu)\mathbf{n}(z/3)\mathbf{g})\psi^\gamma(\boldsymbol{\nu}(\xi)z)\,\mathrm{d}_\alpha z.$$

Changing the variables $z \mapsto \nu z$, we get

$$L^{\alpha,y}_{\beta,\phi}(\mathbf{g},\xi,\nu) = |\alpha^{-1}\nu^2| \int_F \psi^\alpha((\alpha^{-1}\gamma\boldsymbol{\nu}(\xi)\nu - 1)z)\tau^\alpha_{\beta,\phi}(\mathbf{n}(z/3)\mathbf{d}(\nu)\mathbf{g})\,\mathrm{d}_\alpha z.$$

When $\mathbf{g} = \mathrm{diag}(\mathbf{a},\mathbf{a}^{-1})\mathbf{k}$ with $\mathbf{a} \in E^\times$ and $\mathbf{k} \in \mathrm{SL}_2(\mathfrak{o}_E)$, we get

$$L^{\alpha,y}_{\beta,\phi}(\mathbf{g},\xi,\nu) = |\alpha^{-1}\nu^2|\phi(1 - \alpha^{-1}\gamma\boldsymbol{\nu}(\xi)\nu)\beta(\mathbf{a}^2\nu^{-1}).$$

If $\alpha^{-1}\gamma \in \mathcal{B}^c$, then $L^{\alpha,y}_{\beta,\phi}(\mathbf{g},\xi,\nu) = 0$ for $\nu \in \mathcal{B}$ due to our choice of ϕ, which proves (1).

From now on we assume that $\alpha = \gamma$. Recall that $\mathrm{supp}(\beta) \subset \mathcal{B}_E$. Thus $L^{\alpha,y}_{\beta,\phi}(\mathbf{g},\xi,\nu) = 0$ unless $\nu \in \mathcal{B}$, so that the integral over $\mathcal{B}$ can be replaced by the integral over $F^\times$. Changing the variables $\nu \mapsto \boldsymbol{\nu}(\xi)^{-1}\nu$, we get

$$|\boldsymbol{\nu}(\xi)|^2 \int_{\mathcal{B}} \mathcal{W}^\alpha(\mathbf{d}(\nu)\mathbf{g})L^{\alpha,y}_{\beta,\phi}(\mathbf{g},\xi,\nu)\,\frac{\mathrm{d}^\times\nu}{\zeta_F(1)} = |\alpha|^{-1}\mathcal{W}^\alpha_{\beta,\phi}(\mathbf{d}(\boldsymbol{\nu}(\xi)^{-1})\mathbf{g}).$$

Finally, we justify the manipulations above. Our task is to check that

$$\int_{\mathfrak{X}_y}\int_{\mathbf{U}\backslash\mathrm{SL}_2(E)}\int_{\mathcal{B}}\int_F \mathcal{W}^\alpha(\mathbf{d}(\nu)\ddot{\mathbf{g}})\Omega^3_{D,\psi}(\iota(\ddot{\mathbf{g}}))\Phi(\xi y\sigma(\xi)^\iota)$$
$$\times |\nu\boldsymbol{\nu}(\xi)|^2\tau^\alpha_{\beta,\phi}(\mathbf{n}(z/3)\mathbf{d}(\nu)\ddot{\mathbf{g}})\,\mathrm{d}z\frac{\mathrm{d}^\times\nu}{\zeta_F(1)}\mathrm{d}\ddot{\mathbf{g}}\,\mathrm{d}_y\xi$$

is absolutely convergent. We have only to show that the integral

$$\sum_{j=0}^\infty q^{2j}\int_C\int_{E^\times}\int_{\mathrm{SL}_2(\mathfrak{o}_E)}\int_{F^\times}\mathcal{W}^\alpha(\mathbf{d}(\nu)\mathbf{m}(\mathbf{a})\mathbf{k})\Omega^3_{D,\psi}(\iota(\mathbf{m}(\mathbf{a})\mathbf{k}))\Phi(cx_jy\sigma(c)^\iota)$$
$$\times |\nu|^2\beta(\mathbf{a}^2\nu^{-1})|\mathbf{a}|^{-2}\,\mathrm{d}^\times\nu\mathrm{d}\mathbf{k}\mathrm{d}^\times\mathbf{a}\mathrm{d}c$$

is absolutely convergent in view of the relative Cartan decomposition (3.2). Since the integral

$$\int_{F^\times}\int_{E^\times}\int_{F^\times}\frac{|\nu^{-1}\mathbf{a}^2|^{(1-\epsilon)/2}\Phi_0(t\mathbf{a},t^{-1}\mathbf{a})\varphi(t)}{|\nu^{-1}a_1^2|^{\lambda(\pi_1)}|\nu^{-1}a_2^2|^{\lambda(\pi_2)}|\nu^{-1}a_3^2|^{\lambda(\pi_3)}}\,|\nu|^2\beta(\mathbf{a}^2\nu^{-1})\mathrm{d}^\times\nu\mathrm{d}^\times\mathbf{a}\frac{\mathrm{d}^\times t}{|t|^2}$$

is convergent for $\Phi_0 \in \mathcal{S}(E^2)$ and $\varphi \in \mathcal{S}(F)$, the proof is complete. □

For each $n \in \mathbb{N}$ we define $\phi_n \in C^\infty_c(F)$ by

$$\phi_n(z) = |\varpi|^{-n}\phi(\varpi^{-n}z).$$

Then

$$\widehat{(\phi_n)}_\alpha(b) = \hat{\phi}_\alpha(\varpi^n b).$$

The functions ϕ_n satisfy the condition on ϕ.

Take $y \in \mathcal{Y}_D$ with $(y,y)_D = \alpha$. Lemma 4.10(1) gives

$$Z^\star_{\beta,\phi_n}(\mathcal{W}^\alpha,\Phi) = \epsilon(D)I^{\alpha,y}_{\beta,\phi_n}.$$

A function τ_β on $Z_3\mathbf{U}\backslash\mathbf{G}^\star$ is defined by

$$\tau_\beta\left(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}k\right) = \beta(ad^{-1})$$

for $b \in E$, $a, d \in E^\times$ and $k \in \mathrm{GL}_2(\mathfrak{o}_E) \cap \mathbf{G}^\star$. We consider a truncated partial zeta integral defined by

$$Z^\star_\beta(\mathcal{W}^\alpha, \Phi) = \int_{Z_3 U^0 \backslash \mathbf{G}^\star} \mathcal{W}^\alpha(\mathbf{g}) f_\Phi(\eta\iota(\mathbf{g}))\tau_\beta(\mathbf{g})\,\mathrm{d}\mathbf{g}.$$

Since $\lim_{n\to\infty} \tau^\alpha_{\beta,\phi_n}(\mathbf{g}) = \tau_\beta(\mathbf{g})$ for $\mathbf{g} \in \mathbf{G}^\star$, we have

$$\lim_{n\to\infty} Z^\star_{\beta,\phi_n}(\mathcal{W}^\alpha, \Phi) = Z^\star_\beta(\mathcal{W}^\alpha, \Phi)$$

by the dominated convergence theorem. Put

$$\mathcal{W}^\alpha_\beta(\mathbf{g}) = \mathcal{W}^\alpha(\mathbf{g})\tau_\beta(\mathbf{g}), \qquad \phi^\alpha_{\beta,\mathbf{g}}(\nu) = \mathcal{W}^\alpha_\beta(\mathbf{d}(\nu)\mathbf{g})|\nu|.$$

By the proof of Lemma 5.5 of [Ich08]

$$\mathcal{W}^\alpha_{\beta,\phi_n}(\mathbf{g}) = \int_F \phi^\alpha_{\beta,\mathbf{g}}(1-z)\phi_n(z)\,\mathrm{d}_\alpha z$$

for $\mathbf{g} \in \mathbf{G}^\star$. Since $\widehat{(\phi_n)}_\alpha(0) = \hat{\phi}_\alpha(0) = 1$, we arrive at

$$\lim_{n\to\infty} \mathcal{W}^\alpha_{\beta,\phi_n}(\mathbf{g}) = \phi^\alpha_{\beta,\mathbf{g}}(1) = \mathcal{W}^\alpha_\beta(\mathbf{g}).$$

As in the proof of Lemma 5.6 of [Ich08] one can interchange the integrals with the limit as $n \to \infty$ in Lemma 4.10(2), so that

$$\lim_{n\to\infty} I^{\alpha,y}_{\beta,\phi_n} = |\alpha| \int_{\mathrm{GL}_2(K)/K^\times D_y^\times} \int_{\mathbf{U}\backslash \mathrm{SL}_2(E)} H^{D_y}_{\psi^\alpha}(\xi, \ddot{\mathbf{g}}; \Phi_y)\mathcal{W}^\alpha_\beta(\mathbf{d}(\boldsymbol{\nu}(\xi)^{-1})\ddot{\mathbf{g}})\,\mathrm{d}\ddot{\mathbf{g}}\,\mathrm{d}_y\xi.$$

For each $m \in \mathbb{N}$ we choose $\beta_m \in C_c^\infty(E^\times)$, which satisfies the condition on ξ, so that $0 \le \beta_m(\mathbf{a}) \le 1$ and $\lim_{m\to\infty} \beta_m(\mathbf{a}) = 1$ for $\mathbf{a} \in \mathcal{B}_E$. Then $0 \le \tau_{\beta_m}(g) \le 1$ and $\lim_{m\to\infty} \tau_{\beta_m}(g) = 1$ for $g \in \mathbf{G}^\star$. Since $Z^\star(\mathcal{W}^\alpha, \Phi)$ is absolutely convergent, we can use the dominated convergence theorem to interchange the integrals with the limit as $m \to \infty$ to obtain

$$\begin{aligned} Z^\star(\mathcal{W}^\alpha, \Phi) &= \lim_{m\to\infty} Z^\star_{\beta_m}(\mathcal{W}^\alpha, \Phi) \\ &= \epsilon(D) \lim_{m\to\infty}\lim_{n\to\infty} I^{\alpha,y}_{\beta_m,\phi_n} = \epsilon(D)|\alpha| L(2, \epsilon_K)^{-1} \mathscr{I}^\epsilon_D(\mathcal{W}, \Phi) \end{aligned}$$

(cf. Remark 3.1).

5. The basic identity

5.1. **Measures.** We first choose Haar measures on various groups as follows. Once and for all we fix a non-trivial additive character $\psi = \prod_v \psi_v$ of $\mathbb{A}/F$. For each place v of F we take Haar measures $\mathrm{d}^\times z_v = \zeta_{F_v}(1)\frac{\mathrm{d}z_v}{|z_v|}$ of $F_v^\times$ and $\mathrm{d}^\times k_v = \zeta_{K_v}(1)\frac{\mathrm{d}k_v}{\boldsymbol{\alpha}_{K_v}(k_v)}$, where $\mathrm{d}z_v$ and $\mathrm{d}k_v$ are the self-dual measures on F_v and K_v with respect to ψ_v and $\psi_v \circ \mathrm{T}_{K_v/F_v}$, and $\boldsymbol{\alpha}_{K_v}(k_v) = |k_v|_{K_v}$ denotes the normalized absolute value. We define the Tamagawa measure of $\mathbb{A}^\times$ by $\mathrm{d}^\times z = c_F^{-1} \prod_v \mathrm{d}^\times z_v$, where c_F denotes the residue of the complete zeta function $\zeta_F(s) = \prod_v \zeta_{F_v}(s)$ at $s = 1$. Let Ξ' be a gauge form on $D_K^\times$ defined over F. Let $\mathrm{d}\Xi'_v$ be the measure on $D^\times_{K_v}$ associated to Ξ'. On $D^\times_K(\mathbb{A})$ we take the Tamagawa measure

$$\mathrm{d}\Xi' = c_K^{-1}\zeta_K(2)^{-1}D_F^{-4}\prod_v \zeta_{K_v}(1)\zeta_{K_v}(2)\mathrm{d}\Xi'_v.$$

Let $\mathrm{d}\Xi$ be the quotient measure of $\mathrm{d}\Xi'$ by $\mathrm{d}^\times k$.

Let $\mathrm{d}^\times x = \frac{\mathrm{d}x}{(x,x)_D^2}$ be a $D^\times_K$-invariant gauge form on $\mathcal{Y}_D$, where $\mathrm{d}x$ is the differential form $\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4$ on V_D for a system of coordinates x_1, x_2, x_3, x_4 of V_D over F. Recall the map $\wp = \wp_1 : D^\times_K \to \mathcal{Y}_D$ defined by $\wp(\xi) = \xi\sigma(\xi)^\iota$ for $\xi \in D^\times_K$, and the subgroup $\mathcal{U} = \mathcal{U}_1 = \{\xi \in D^\times_K \mid \wp(\xi) = 1\}$. Let $\mu = \Xi'/\wp^*\mathrm{d}^\times x$ be the gauge form on $\mathcal{U}$ determined by Ξ' and $\wp^*\mathrm{d}^\times x$ (see p. 12 of [Wei65]). We define the Tamagawa measure on $\mathcal{U}(\mathbb{A})$ by

$$\mathrm{d}\mu = \zeta_F(2)^{-1}D_F^{-2}\prod_v \zeta_{F_v}(2)\mathrm{d}\mu_v.$$

Let $\mathrm{d}\mathbf{g}''$, $\mathrm{d}\mathbf{g}'$ and $\mathrm{d}h$ be the Tamagawa measures on $Z_3(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})$, $\mathrm{SL}_2(\mathbb{E})$ and $\mathrm{O}(V_D,\mathbb{A})$, respectively.

5.2. **Siegel Eisenstein series.** Fix a maximal compact subgroup $\mathcal{K}$ of $\mathrm{GSp}_6(\mathbb{A})$. Let $I_3(s,\epsilon_K) = \otimes_v I_3(s,\epsilon_{K_v})$ consist of all right $\mathcal{K}$-finite functions $f^{(s)} : \mathrm{GSp}_6(\mathbb{A}) \to \mathbb{C}$ such that

$$f^{(s)}(\mathbf{d}(t)\mathbf{n}(z)\mathbf{m}(a)g) = \epsilon_K(\det a)|t^{-3}(\det a)^2|^{s+1}f^{(s)}(g)$$

for $t \in \mathbb{A}^\times$, $z \in \mathrm{Sym}_3(\mathbb{A})$, $a \in \mathrm{GL}_3(\mathbb{A})$ and $g \in \mathrm{GSp}_6(\mathbb{A})$. The Eisenstein series associated to $f^{(s)} \in I_3(s,\epsilon_K)$ is defined by

$$E(g; f^{(s)}) = \sum_{\gamma\in P_3(F)\backslash\mathrm{GSp}_6(F)} f^{(s)}(\gamma g)$$

for $\Re s > 2$. The series has meromorphic continuation to the whole s-plane and has no poles on the unitary axis $\Re s = 0$.

Define the character δ_{P_3} of $P_3(\mathbb{A})$ by $\delta_{P_3}(\mathbf{d}(t)\mathbf{n}(z)\mathbf{m}(a)) = |t|^{-3}|\det a|^2$. We extend δ_{P_3} to the right $\mathcal{K}$-invariant function on $\mathrm{GSp}_6(\mathbb{A})$ by the Iwasawa decomposition. Let $\Phi = \otimes_v\Phi_v \in \mathcal{S}(V_D^3(\mathbb{A}))$. Set $f_\Phi(g) = \prod_v f_{\Phi_v}(g_v)$ for $g = (g_v) \in \mathrm{GSp}_6(\mathbb{A})^\star$. Recall that f_Φ is extended to a section $f_\Phi^{(s)}$ of $I_3(s,\epsilon_K)$ by $f^{(s)}_\Phi(g) = \delta_{P_3}(g)^s f^{(0)}_\Phi(g)$ for $g \in \mathrm{GSp}_6(\mathbb{A})$.

We associated to Φ the series defined for $\Re s > 2$ and $g \in \mathrm{GSp}_6(\mathbb{A})^\star$ by

$$\mathcal{E}(s, g; \Phi) = \sum_{\gamma\in P_3(F)^\star\backslash\mathrm{GSp}_6(F)^\star} \delta_{P_3}(\gamma g)^s f_\Phi(\gamma g).$$

We extend $\mathcal{E}(s, g; \Phi)$ to a left $\mathrm{GSp}_6(F)$-invariant function on

$$\mathcal{G}_K = \mathrm{GSp}_6(F)\mathrm{GSp}_6(\mathbb{A})^\star = \{g \in \mathrm{GSp}_6(\mathbb{A}) \mid \nu_3(g) \in F^\times\mathbb{B}\}.$$

The subgroup $\mathbb{B}$ of $\mathbb{A}^\times$ is defined in Definition 2.7. The set $\mathcal{G}_K$ is a subgroup of $\mathrm{GSp}_6(\mathbb{A})$ of index 2 as $F^\times\mathbb{B} = F^\times\mathrm{N}_{K/F}(\mathbb{K}^\times)$. The series $\mathcal{E}(s, g; \Phi)$ is related to $E(g; f^{(s)}_\Phi)$ in the following way:

Proposition 5.1. *(1) If $g \in \mathcal{G}_K$, then $E(g; f_\Phi^{(s)}) = \mathcal{E}(s, g; \Phi)$.*

(2) If $g \notin \mathcal{G}_K$, then $\lim_{s \to 0} E(g; f_\Phi^{(s)}) = 0$.

Proof. The first statement is clear from $\mathrm{GSp}_6(F) = P_3(F)\mathrm{GSp}_6(F)^\star$. Suppose that $\Phi = \otimes_v \Phi_v$ is factorizable. Take an idèle $a \notin F^\times \mathbb{B}$. Note that

$$f_\Phi^{(0)}(g\mathbf{d}(a)) = |a|^{-3}\Omega^3_{D,\psi}(\mathbf{d}(a)^{-1}g\mathbf{d}(a))\Phi(0)$$

for $g \in \mathrm{Sp}_6(\mathbb{A})$. Then

$$\Omega^3_{D_v,\psi_v}(\mathbf{d}(a_v)^{-1}g_v\mathbf{d}(a_v)) = \Omega^3_{D_v,\psi_v^{a_v}}(g_v)$$

is the local Weil representation associated to the dual pair $\mathrm{Sp}_6(F_v) \times \mathrm{O}(V^{a_v}_{D_v})$. By Remark 2.4 there exists no global quadratic space with $V^{a_v}_{D_v}$ as its completions. In other words, the series $E(g\mathbf{d}(a); f_\Phi^{(s)})$ is incoherent and vanishes at $s = 0$ by Theorem 3.1(ii) of [KR94]. □

5.3. **The Siegel-Weil formula.** When D_K is not split, the theta integral is defined, for $g \in \mathrm{GSp}_6(\mathbb{A})^\star$ and $\Phi \in \mathcal{S}(V_D^3(\mathbb{A}))$, by

$$\theta(g; \Phi) = \int_{\mathrm{O}(V_D,F)\backslash \mathrm{O}(V_D,\mathbb{A})} \Theta(hh', g; \Phi)\,\mathrm{d}h,$$

where $h' \in \mathrm{GO}(V_D, \mathbb{A})$ with $\boldsymbol{\nu}(h') = \nu_3(g)$. It does not depend on the choice of h'. Here the Haar measure $\mathrm{d}h$ gives $\mathrm{O}(V_D, F)\backslash\mathrm{O}(V_D, \mathbb{A})$ volume 1. In the case $D_K \simeq \mathrm{M}_2(K)$ the theta integral can be defined by regularization (see [KR94]). The group $\mathcal{B} = \mathbb{B} \cap F^\times$ consists of idèles $\boldsymbol{\nu}(h)$ with $h \in \mathrm{GO}(V_D, F)$ by Eichler's norm theorem. It follows from Remark 2.6 that

$$\theta(z\gamma g; \Phi) = \epsilon_K(z)\theta(g; \Phi)$$

for $z \in Z_3(\mathbb{A})$, $\gamma \in \mathrm{GSp}_6(F)^\star$ and $g \in \mathrm{GSp}_6(\mathbb{A})^\star$.

The Siegel-Weil formula is now stated as follows:

$$\mathcal{E}(0, g; \Phi) = 2\theta(g; \Phi).$$

The reader who has interested in this formula can consult [HK91, Theorem 4.1] or [KR94, Theorem 6.12].

5.4. **The seesaw identity.** Put $\mathbb{E} = \mathbb{A} \times \mathbb{A} \times \mathbb{A}$. Let Π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{E})$ whose central character has the restriction ϵ_K to $\mathbb{A}^\times$. For a cusp form $\mathcal{F} = f_1 \otimes f_2 \otimes f_3 \in \Pi$ and $\Phi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \in \mathcal{S}(V_D^3(\mathbb{A}))$ the global zeta integral is defined by

$$Z(\mathcal{F}, f^{(s)}) = \int_{Z_3(\mathbb{A})\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})} \mathcal{F}(\mathbf{g}'')E(\iota(\mathbf{g}''); f^{(s)})\,\mathrm{d}\mathbf{g}'',$$

where $\mathrm{d}\mathbf{g}''$ is the Tamagawa measure of $Z_3(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})$.

We consider the period integral I, which is an element of

$$\mathrm{Hom}_{D_K^\times(\mathbb{A})}(\pi^D_{1,K} \otimes \pi^D_{2,K} \otimes \pi^D_{3,K}, \mathbb{C}),$$

defined by

$$I(\phi_1 \otimes \phi_2 \otimes \phi_3) = \int_{\mathbb{K}^\times D_K^\times(F)\backslash D_K^\times(\mathbb{A})} \phi_1(\xi)\phi_2(\xi)\phi_3(\xi)\,\mathrm{d}\xi,$$

where $\mathrm{d}\xi$ is the Tamagawa measure on $\mathbb{K}^\times\backslash D_K^\times(\mathbb{A})$. Put

$$\mathbf{H} = \{(h_1, h_2, h_3) \in \mathrm{GO}(V_D)^3 \mid \boldsymbol{\nu}(h_1) = \boldsymbol{\nu}(h_2) = \boldsymbol{\nu}(h_3)\},$$
$$\mathbf{G}(\mathbb{A})^\star = \mathbf{G}(\mathbb{A}) \cap \mathrm{GSp}_6(\mathbb{A})^\star.$$

Let $\mathcal{F} = f_1 \otimes f_2 \otimes f_3 \in \Pi$ and $\Phi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \in \mathcal{S}(V_D^3(\mathbb{A}))$. We write

$$\theta(h; \mathcal{F}, \Phi) = \prod_{i=1}^{3} \theta(h_i; f_i, \varphi_i) = \int_{\mathrm{SL}_2(E)\backslash \mathrm{SL}_2(\mathbb{E})} \mathcal{F}(\mathbf{g}')\Theta(h; \iota(\mathbf{g}'\mathbf{g}_h), \Phi)\,\mathrm{d}\mathbf{g}'$$

for $h = (h_1, h_2, h_3) \in \mathbf{H}(\mathbb{A})$, where $\mathbf{g}_h \in \mathbf{G}(\mathbb{A})$ with $\det(\mathbf{g}_h) = \boldsymbol{\nu}(h)$ and $\mathrm{d}\mathbf{g}'$ denotes the Tamagawa measure of $\mathrm{SL}_2(\mathbb{E})$.

Proposition 5.2 (The seesaw identity)**.**

$$\lim_{s\to 0} Z(\mathcal{F}, f_\Phi^{(s)}) = I(\theta(\mathcal{F}, \Phi)).$$

Proof. Let $\mathbf{G}_K = \mathbf{G}(F)\mathbf{G}(\mathbb{A})^\star = \mathbf{G}(\mathbb{A}) \cap \mathcal{G}_K$ be a subgroup of $\mathbf{G}(\mathbb{A})$ of index 2. Since the function $\mathbf{g} \mapsto E(\iota(\mathbf{g}); f^{(s)})$ is the extension of $\mathcal{E}(0, \iota(\mathbf{g}); \Phi)$ by zero from $\mathbf{G}_K$ to $\mathbf{G}(\mathbb{A})$ by Proposition 5.1,

$$\lim_{s\to 0} Z(\mathcal{F}, f_\Phi^{(s)}) = \int_{Z_3(\mathbb{A})\mathbf{G}(F)\backslash \mathbf{G}_K} \mathcal{F}(\mathbf{g}'')\mathcal{E}(0, \iota(\mathbf{g}''); \Phi)\,\mathrm{d}\mathbf{g}''.$$

The Siegel-Weil formula gives

$$\lim_{s\to 0} Z(\mathcal{F}, f_\Phi^{(s)}) = 2\int_{Z_3(\mathbb{A})\mathbf{G}(F)^\star\backslash \mathbf{G}(\mathbb{A})^\star} \mathcal{F}(\mathbf{g}'')\theta(\iota(\mathbf{g}''); \Phi)\,\mathrm{d}\mathbf{g}'',$$

where $\mathbf{G}(F)^\star = \mathbf{G}(F) \cap \mathbf{G}(\mathbb{A})^\star$. Since $Z_3\backslash\mathbf{G} \simeq \mathrm{PGL}_2 \ltimes (\mathrm{SL}_2 \times \mathrm{SL}_2)$, the Tamagawa measure $\mathrm{d}\mathbf{g}''$ gives the domain $Z_3(\mathbb{A})\mathbf{G}(F)^\star\backslash\mathbf{G}(\mathbb{A})^\star$ volume 1.

Now we apply the seesaw pair:

$$\begin{array}{cccccc} \theta(h;\mathcal{F},\Phi) & \mathbf{H} & & & \mathrm{GSp}_6^\star & \theta(g;\Phi) \\ & | & \diagdown & \diagup & | & \\ \mathbb{1} & \mathrm{GO}(V_D) & & & \mathbf{G}^\star & \mathcal{F} \end{array}$$

Set $C := \mathcal{B}\mathbb{A}^{\times 2}\backslash\mathbb{B}$. Note that

$$Z_3(\mathbb{A})\mathbf{G}(F)^\star\mathrm{SL}_2(\mathbb{E})\backslash\mathbf{G}(\mathbb{A})^\star \simeq Z_D(\mathbb{A})\mathrm{GO}(V_D, F)\mathrm{O}(V_D, \mathbb{A})\backslash\mathrm{GO}(V_D, \mathbb{A}) \simeq C$$

is compact. Fix a Haar measure $\mathrm{d}c$ giving C volume 1. We have

$$\begin{aligned} &\int_{Z_3(\mathbb{A})\mathbf{G}(F)^\star\backslash\mathbf{G}(\mathbb{A})^\star} \mathcal{F}(\mathbf{g}'')\theta(\iota(\mathbf{g}''); \Phi)\,\mathrm{d}\mathbf{g}'' \\ &= \int_C \int_{\mathrm{SL}_2(E)\backslash\mathrm{SL}_2(\mathbb{E})} \mathcal{F}(\mathbf{g}'\mathbf{g}_c) \int_{\mathrm{O}(V_D,F)\backslash\mathrm{O}(V_D,\mathbb{A})} \Theta(hh_c, \iota(\mathbf{g}'\mathbf{g}_c); \Phi)\,\mathrm{d}h\mathrm{d}\mathbf{g}'\mathrm{d}c \end{aligned}$$

$$=\int_C\int_{\mathrm{O}(V_D,F)\backslash \mathrm{O}(V_D,\mathbb{A})}\theta(hh_c;\mathcal{F},\Phi)\,\mathrm{d}h\mathrm{d}c$$
$$=\int_{Z_D(\mathbb{A})\mathrm{GO}(V_D,F)\backslash \mathrm{GO}(V_D,\mathbb{A})}\theta(h;\mathcal{F},\Phi)\,\mathrm{d}h.$$

This integral factorizes into the product of local invariant trilinear forms constructed in Section 3 by Prasad's uniqueness theorem. Put

$$\tilde{H}=(\mathbb{G}_m\times D_K^\times)\rtimes\langle\mathbf{t}\rangle,\qquad \tilde{H}^0=\mathbb{G}_m\times D_K^\times,\qquad Z_{\tilde{H}^0}=\mathbb{G}_m\times \mathrm{R}_{K/F}\mathbb{G}_m.$$

Recall the homomorphism $\rho:\tilde{H}\to\mathrm{GO}(V_D)$ defined in (2.1). Since $\mathrm{d}h$ gives

$$Z_D(\mathbb{A})\mathrm{GO}(V_D,F)\backslash\mathrm{GO}(V_D,\mathbb{A})$$

volume 1, we have

$$2\int_{Z_{\tilde{H}^0}(\mathbb{A})\tilde{H}(F)\backslash\tilde{H}(\mathbb{A})}\theta(\rho(h);\mathcal{F},\Phi)\,\mathrm{d}h$$
$$=2\int_{Z_{\tilde{H}^0}(\mathbb{A})\tilde{H}^0(F)\backslash\tilde{H}^0(\mathbb{A})}\theta(\rho(h);\mathcal{F},\Phi)\,\mathrm{d}h$$
$$=\int_{\mathbb{K}^\times D_K^\times(F)\backslash D_K^\times(\mathbb{A})}\theta(\xi;\mathcal{F},\Phi)\,\mathrm{d}\xi.$$

If v is inert, then $D_{K_v}^\times$-invariant trilinear forms are invariant under the action of $\mathbf{t}$ by Lemma 3.7. The analogous invariance holds for split places. Therefore the integral over $Z_{\tilde{H}^0}(\mathbb{A})\tilde{H}(F)\backslash\tilde{H}(\mathbb{A})$ can be replaced by the integral over $Z_{\tilde{H}^0}(\mathbb{A})\tilde{H}^0(F)\backslash\tilde{H}^0(\mathbb{A})$ in the second line. □

5.5. **The proof of Theorem 1.3.** We hereafter require the base change Π_K to be cuspidal. We write $L(s,\Pi_v)$ for the triple product L-factor of Π_v. The epsilon factor is defined by the relation

$$\varepsilon(s,\Pi_v,\psi_v)=\gamma(s,\Pi_v,\psi_v)\frac{L(s,\Pi_v)}{L(1-s,\Pi_v^\vee)}.$$

Clearly, $\varepsilon(\Pi_v)=\gamma(\Pi_v)$ if Π_v is self-dual.

For a quaternion algebra D over F we consider the following condition:

$$(\sharp)\qquad \epsilon(D_v)\neq-\epsilon_K(-1)\gamma(\Pi_v)\text{ for all }v.$$

Proposition 5.3. *(1) If D satisfies ($\sharp$), then it satisfies (JL) and (Per).*

(2) If there exists a place v such that $\gamma(\Pi_v)^2\neq1$, then there is a quaternion algebra which satisfies ($\sharp$).

Proof. Since $D_{K_v}\simeq\mathrm{M}_2(K_v)$ unless $K_v\simeq F_v\times F_v$, the Jacquet-Langlands lift π_K^D exists if and only if the local Jacquet-Langlands lift π_v^D of π_v to $D_v^\times$ exists for all the split places v.

Assume that D satisfies ($\sharp$). Then the functional $\mathscr{I}_{D_v}^+$ is non-vanishing for all v by Theorem 1.1. A fortiori, $\pi_v^{D_v}$ exists and $B_{1,v},B_{2,v},B_{3,v}$ are non-vanishing. Thus π_K^D exists. For B_i to be non-vanishing there is no global obstruction in view of Propositions 2.10 and 2.11(2). Hence ($\sharp$) implies (Per).

If the cardinality of the residue field of $\mathfrak{o}_{F_v}$ is sufficiently large, then $L\big(\frac{1}{2},\Pi_v\big)\neq -L\big(\frac{1}{2},\Pi_v^\vee\big)$ in view of Remark 3.4(2). It follows that $\gamma(\Pi_v)\neq -1$ for all but finitely many places of F. One can now trivially prove (2) by the Minkowski-Hasse theorem. □

We are now ready to prove the central value formula. From now on we assume that D satisfies (♯). We denote the Jacquet-Langlands lift of Π_K to $(D_K\otimes\mathbb{E})^\times$ by Π_K^D. Take $\varphi=\phi_1\otimes\phi_2\otimes\phi_3=\otimes_v\varphi_v\in\Pi_K^D$ so that

$$\mathscr{B}(\varphi):=B_1(\phi_1)B_2(\phi_2)B_3(\phi_3)\neq 0.$$

Recall the functionals $B_{i,v}^\natural$ and $I_v^\natural$ defined in §4.4. By by Remark 3.4(2) and Lemma 3.5 $I_v^\natural$ makes sense. Set $\mathscr{B}_v^\natural=B_{1,v}^\natural\otimes B_{2,v}^\natural\otimes B_{3,v}^\natural$. The formula stated in Theorem 1.3 is equivalent to the following formula:

$$\frac{I(\varphi)}{\mathscr{B}(\varphi)}=2^{-3}\cdot\frac{\zeta_F(2)^2L\big(\frac{1}{2},\Pi\big)}{L(1,\mathrm{Ad}(\Pi)\otimes\epsilon_K)}\cdot\prod_v\frac{I_v^\sharp(\varphi_v)}{\mathscr{B}_v^\natural(\varphi_v)}.$$

If $\mathcal{F}$ has the factorizable Whittaker function $\mathcal{W}=\otimes_v\mathcal{W}_v$ with respect to $\bar\psi$ and if $\Phi=\otimes_v\Phi_v$ is factorizable, then

$$Z(\mathcal{F},f_\Phi^{(s)})=\prod_v Z(\mathcal{W}_v,f_{\Phi_v}^{(s)})=\frac{L^S\big(s+\frac{1}{2},\Pi\big)}{L^S(2s+2,\epsilon_K)\zeta_F^S(4s+2)}\prod_{v\in S}Z(\mathcal{W}_v,f_{\Phi_v}^{(s)}),$$

where $S=S_{f_1,\varphi_1}\cup S_{f_2,\varphi_2}\cup S_{f_3,\varphi_3}$. Take $\mathcal{F}\in\Pi$ and $\Phi=\otimes_v\Phi_v\in\mathcal{S}(V_D^3(\mathbb{A}))$ so that $\theta(\mathcal{F},\Phi)=\varphi$. Let $\mathcal{W}=\prod_v\mathcal{W}_v$ be the Whittaker function of $\mathcal{F}$ with respect to $\bar\psi$. The formula (4.6) remains true at split places of F (cf. p. 296 of [Ich08]). Hence Proposition 5.2 gives

$$\begin{aligned}I(\theta(\mathcal{F},\Phi))&=\frac{L^S\big(\frac{1}{2},\Pi\big)}{L^S(2,\epsilon_K)\zeta_F^S(2)}\prod_{v\in S}Z(\mathcal{W}_v,f_{\Phi_v}^{(0)})\\&=\zeta_F(2)^{-1}L\left(\frac{1}{2},\Pi\right)\prod_{v\in S}I_v^\natural(\theta(\mathcal{W}_v,\Phi_v)).\end{aligned}$$

Since

$$\mathscr{B}(\varphi)=\frac{2^3}{\zeta_F(2)^3}\cdot L(1,\mathrm{Ad}(\Pi)\otimes\epsilon_K)\cdot\prod_v\mathscr{B}_v^\natural(\varphi_v)$$

by Proposition 2.11, we have thus completed our proof.

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