THE ALGEBRAIC FUNCTIONAL EQUATION OF SELMER GROUPS FOR CM FIELDS

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Abstract. Because the analytic functional equation holds for Katz p-adic L-function for CM fields, the algebraic functional equation of the Selmer groups for CM fields is expected to hold. In this note we prove it following the specialization principle developed by T. Ochiai.

1. Introduction

The aim of this note is to prove the algebraic functional equation of the Selmer groups for CM fields which is predicted by the analytic functional equation for Katz p-adic L-function and Iwasawa main conjecture for CM fields. The idea is to use the specialization principle developed by T. Ochiai in [Och05].

Let us briefly recall Iwasawa main conjecture for CM fields. Let $\mathcal{F}$ be a totally real subfield of degree d over $\mathbf{Q}$ and $\mathcal{K}$ be a totally imaginary quadratic extension of $\mathcal{F}$ and let c be the complex conjugation, the unique nontrivial element in $\mathrm{Gal}(\mathcal{K}/\mathcal{F})$. Let p be an odd rational prime. The main conjecture for CM fields states equality of two ideals generated by p-adic L-functions and the characteristic power series of Selmer groups respectively. To introduce them, we make the ordinary assumption (Ord) as follows:

(Ord) every prime of $\mathcal{F}$ above p splits in $\mathcal{K}$.

Let S_p be the set of places of $\mathcal{K}$ above p. Then (Ord) is equivalent to the existence of a p-adic CM type Σ which is a subset in S_p such that Σ and its complex conjugation Σ^c form a partition of S_p. Namely

$$\Sigma \cap \Sigma^c = \emptyset,\ \Sigma \bigsqcup \Sigma^c = S_p.$$

By definition Σ^c is also a p-adic CM type.

Let $\mathcal{K}^+_\infty$ be the cyclotomic $\mathbf{Z}_p$-extension and $\mathcal{K}^-_\infty$ be the anticyclotomic $\mathbf{Z}_p^d$-extensions of $\mathcal{K}$ respectively. Let $\mathcal{K}_\infty = \mathcal{K}^+_\infty\mathcal{K}^-_\infty$. If one assumes Leopoldt's conjecture, then $\mathcal{K}_\infty$ would be the composition of all $\mathbf{Z}_p$-extensions of $\mathcal{K}$. Let $\Gamma := \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$ be a free $\mathbf{Z}_p$-module of rank $1+d$. Let $\mathcal{K}'$ be a finite abelian extension of $\mathcal{K}$ which is linearly disjoint from $\mathcal{K}_\infty$ and $\Delta = \mathrm{Gal}(\mathcal{K}'/\mathcal{K})$. Let $\mathcal{K}'_\infty = \mathcal{K}'\mathcal{K}_\infty$ and $\mathfrak{G} = \mathrm{Gal}(\mathcal{K}'_\infty/\mathcal{K}) \simeq \Delta\times\Gamma$. We let $\psi : \mathfrak{G} \to \mathbf{C}_p^\times$ be a continuous p-adic character. Let D_w be the decomposition group of a place w. We further assume that $\psi|_{D_w}$ for all $w|p$ are locally algebraic. Let $\mathcal{O}_\psi = \mathbf{Z}_p[\mathrm{Im}\psi]$ be the ring of values of ψ and let $\mathcal{O}$ be a complete discrete valuation ring which is finite flat over $\mathcal{O}_\psi$.

1.1. p-adic L-functions and Selmer groups for CM fields.

We shall formulate the main conjecture for CM fields from the p-adic Galois representation point of view [Gre94]. We begin with some notation. Let $G_{\mathcal{K}} = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathcal{K})$ be the absolute Galois group of $\mathcal{K}$ and $\Lambda = \mathcal{O}[\![\Gamma]\!]$. Then Λ is an Iwasawa algebra in $d+1$-variables. Define $\Psi : G_{\mathcal{K}} \to \Lambda^\times$ the universal Λ-adic character associated to ψ by

$$\begin{aligned}\Psi : G_{\mathcal{K}} &\longrightarrow \Lambda^\times\\ g &\longrightarrow \psi(g)g|_{\mathcal{K}_\infty}.\end{aligned}$$

On the analytic side, one has the p-adic L-function for CM fields, $L_p(\Psi,\Sigma) \in \Lambda$, which is constructed by Katz [Kat78] if the conductor of ψ divides p^∞ and by Hida and Tilouine [HT93] in general. Roughly $L_p(\Psi,\Sigma)$ interpolates Hecke L-values for $\mathcal{K}$ p-adically. Moreover $L_p(\Psi,\Sigma)$ satisfies a functional equation (*cf.* [HT93, Theorem 2]).

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On the algebraic side, one has the Selmer group for CM fields. We recall its definition after introducing some notation. For a locally compact topological abelian group M, we denote by M^* the Pontryagin dual of M. Then $\Lambda^* = \mathrm{Hom}_{cont}(\Lambda, \mathbf{Q}_p/\mathbf{Z}_p)$ has a natural discrete Λ-module structure given by $a \cdot f(x) := f(xa)$, $a, x \in \Lambda$. Let w be a place of $\mathcal{K}$. We write I_w for the inertia group at w and denote by F_w a *geometric* Frobenius in D_w. Let $\mathfrak{c}$ be the prime-to-p conductor of ψ and S_ψ be the set of finite places dividing $\mathfrak{c}$. Let $S \supset S_\psi$ be a finite set of prime-to-p places of $\mathcal{K}$ and let $\mathcal{K}_S$ be the maximal $S \cup S_p$-ramified algebraic extension of $\mathcal{K}$. Define the Selmer group $Sel_{\mathcal{K}}(\varPsi, \varSigma)$ associated to $(\varPsi, \varSigma)$ by

$$Sel_{\mathcal{K}}(\varPsi, \varSigma) = \ker\left\{ H^1(\mathcal{K}_S/\mathcal{K}, \varPsi \otimes \Lambda^*) \to \prod_{w \in S \coprod \varSigma^c} H^1(I_w, \varPsi \otimes \Lambda^*) \right\}, \tag{1.1}$$

It follows from [HT94, Theorem 1.2.2] that $Sel_{\mathcal{K}}(\varPsi, \varSigma)$ is a cofinitely generated and cotorsion discrete Λ-module. Let $F(\varPsi, \varSigma)$ denote the characteristic power series of $Sel_{\mathcal{K}}(\varPsi, \varSigma)$ which is unique up to Λ-units (See Def. 2.1 (1)). Let $\mathrm{ht}_1(\Lambda)$ be the set of height one primes of Λ. For $P \in \mathrm{ht}_1(\Lambda)$, we let ord_P be the valuation at P. Then Iwasawa main conjecture for CM fields is stated as follows.

Conjecture 1 (Iwasawa main conjecture for CM fields)**.** *For every $P \in \mathrm{ht}_1(\Lambda)$, we have*

$$\mathrm{ord}_P(L_p(\varPsi, \varSigma)) = \mathrm{ord}_P(F(\varPsi, \varSigma)).$$

We also define the non-primitive p-adic L-function $L_p^S(\varPsi, \varSigma)$ by

$$L_p^S(\varPsi, \varSigma) = L_p(\varPsi, \varSigma) \cdot \prod_{w \in S \setminus S_\psi} (1 - \varPsi(F_w)).$$

Similarly, the non-primitive Selmer group is defined by

$$Sel_{\mathcal{K}}^S(\varPsi, \varSigma) = \ker\left\{ H^1(\mathcal{K}_S/\mathcal{K}, \varPsi \otimes \Lambda^*) \to \prod_{w \in S_\psi \coprod \varSigma^c} H^1(I_w, \varPsi \otimes \Lambda^*) \right\}. \tag{1.2}$$

Let $F^S(\varPsi, \varSigma)$ be the characteristic power series of $Sel_{\mathcal{K}}^S(\varPsi, \varSigma)$. We also consider the *dual* version of the main conjecture for CM fields which has the advantage of including non-primitive p-adic L-functions and Selmer groups. In the case of main conjecture for totally real fields, such a dual version is proposed by R. Greenberg in [Gre77].

Let ε be the p-adic cyclotomic character of $G_{\mathcal{K}}$ and $\omega : G_{\mathcal{K}} \to \mu_{p-1}$ be the p-adic Teichmüller character. Define the *Cartier* dual character $\varPsi^D$ of $\varPsi$ by $\varPsi^D = \varPsi^{-1}\varepsilon$. Then the dual version of the main conjecture for CM fields is stated as follows.

Conjecture 2. *For every $P \in \mathrm{ht}_1(\Lambda)$, we have*

$$\mathrm{ord}_P(L_p^S(\varPsi, \varSigma)) = \mathrm{ord}_P(F^S(\varPsi^D, \varSigma^c)).$$

1.2. **Main result.** Our main result is as follows.

Theorem 1.1 (Algebraic functional equation)**.** *For every $P \in \mathrm{ht}_1(\Lambda)$, we have*

$$\mathrm{ord}_P(F(\varPsi, \varSigma)) = \mathrm{ord}_P(F(\varPsi^D, \varSigma^c)).$$

Remark 1.

(1) Theorem 1.1 is an immediate consequence of the main conjecture for CM fields (Conjecture 1) combined with the functional equation of Katz p-adic L-functions.
(2) The general functional equation of the Selmer groups associated to the cyclotomic deformation of p-adic Galois representations is proved by R. Greenberg [Gre89].

This theorem has the following corollary.

Corollary 1.2. *If $\psi|_\Delta \neq 1$, Conjecture 1 is equivalent to Conjecture 2.*

2. The proof

2.1. Notation and definitions. We first prepare some notation and definitions.

Definition 2.1. Let R be a compact normal Noetherian domain and $\mathrm{ht}_1(R)$ be the set of height one primes of R. For $P \in \mathrm{ht}_1(R)$, let R_P be the localization of R at P. Let $\mathcal{S}$ be a cofinitely generated R-module and let $\mathcal{S}^*$ be the Pontryagin dual of $\mathcal{S}$.

(1) If $\mathcal{S}$ is cotorsion, Define the characteristic ideal $\mathrm{char}_R\mathcal{S}$ by

$$\mathrm{char}_R\mathcal{S} = \prod_{P\in\mathrm{ht}_1(R)} P^{\ell_P(\mathcal{S})},$$

where $\ell_P(\mathcal{S}) = \mathrm{length}_{R_P}(\mathcal{S}^* \otimes_R R_P)$. The characteristic power series is a generator of $\mathrm{char}_R\mathcal{S}$.

(2) $\mathcal{S}$ is said to be pseudo-null if $\mathcal{S}^*$ is a pseudo-null R-module.

(3) If $\mathcal{S}$ is a finite $\mathbf{Z}_p$-module, we put

$$\ell_p(\mathcal{S}) = \mathrm{length}_{\mathbf{Z}_p}(\mathcal{S}).$$

(4) Denote by $\mathcal{S}_{null}$ the maximal pseudo-null R-module quotient of $\mathcal{S}$.

(5) For $\mathcal{S}$ and $\mathcal{S}'$ two discrete cofinitely generated R-modules, we say $\mathcal{S} \sim \mathcal{S}'$ if there exists a R-module morphism $\mathcal{S} \to \mathcal{S}'$ such that the kernel and cokernel are pseudo-null.

The following observation is useful.

Lemma 2.2. *Suppose that R is UFD and $\mathcal{S}$ is cotorsion. Let $f \in R$ be prime to $\mathrm{char}_R\mathcal{S}$. Then*

$$\mathcal{S} \otimes_R R/(f) \overset{\sim}{\to} \mathcal{S}_{null} \otimes_R R/(f).$$

In particular if $R \simeq \mathcal{O}[\![T]\!]$, $\ell_p(\mathcal{S} \otimes_R R/(f))$ is uniformly bounded for all $f \in R$ prime to $\mathrm{char}_R\mathcal{S}$.

Proof. The Pontryagin dual $\mathcal{S}^*$ of $\mathcal{S}$ is a finitely generated torsion R-module. By the structure theorem for finitely generated torsion modules over a normal domain (*cf.* [Bou65, §4.4, THÉORÈM 5, p.253]), there exist R-module morphisms

$$K \hookrightarrow \mathcal{S}^* \to E \twoheadrightarrow C,$$

where K and C are pseudo-null and $E \overset{\sim}{\to} \oplus_i R/(g_i)$. Note that by definition the Pontryagin dual K^* of K is $\mathcal{S}_{null}$ and $\mathrm{char}_R\mathcal{S} = (\prod_i g_i)$. Thus if f is prime to $\mathrm{char}_R\mathcal{S}$, then $K[f] = \mathcal{S}^*[f]$ and hence $\mathcal{S} \otimes_R R/(f) \overset{\sim}{\to} \mathcal{S}_{null} \otimes_R R/(f)$. □

Definition 2.3. Put $G_S = \mathrm{Gal}(\mathcal{K}_S/\mathcal{K})$ and $G = G_{S_\psi}$ for brevity. Let A be a discrete G_S-module and $\mathscr{L}_w$ be a Λ-submodule in $H^1(D_w, A)$ for each $w \in S \cup S_p$. Then the Selmer group $H^1_{\mathscr{L}}(A)$ associated to the local condition $\mathscr{L} = \{\mathscr{L}_w\}_{w\in S\cup S_p}$ is defined by

$$H^1_{\mathscr{L}}(A) = \ker\left\{ H^1(G_S, A) \to \prod_{w\in S\cup S_p} \frac{H^1(D_w, A)}{\mathscr{L}_w} \right\}.$$

Define the local condition $\mathscr{L}(\Sigma) = \{\mathscr{L}(\Sigma)_w\}_{w\in S\cup S_p}$ by

$$\mathscr{L}(\Sigma)_w = \begin{cases} H^1(D_w, A) & , w \in \Sigma \\ 0 & , w \in \Sigma^c \\ H^1(D_w/I_w, A^{I_w}) & , w \in S \end{cases}. \tag{2.1}$$

We define the strict Selmer group $Sel^{\mathrm{str}}_{\mathcal{K}}(\Psi, \Sigma)$ by

$$Sel^{\mathrm{str}}_{\mathcal{K}}(\Psi, \Sigma) := H^1_{\mathscr{L}(\Sigma)}(\Psi \otimes \Lambda^*).$$

This definition is independent of the choice of $S \supset S_\psi$.

If A is a finite $\mathbf{Z}_p$-module, we set $h^1_{\mathscr{L}}(A) = \ell_p(H^1_{\mathscr{L}}(A))$,

$$h^i(A) = \ell_p(H^i(G, A)) \text{ and } h^i_\Sigma(A) = \sum_{w\in\Sigma} \ell_p(H^1(D_w, A))$$

for $i = 0, 1, 2$. We write $\chi(G, A)$ (resp. $\chi(D_w, A)$) for the global (resp. local) Euler-characteristics.

Lemma 2.4. *Let $A = \Psi \otimes R^*$ for a finite quotient ring R of Λ and let $A^D = \Psi^D \otimes R^*$ be the Cartier dual of A. We have the following long exact sequence.*

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathscr{L}}(A) \arrow[r] & H^1(G,A) \arrow[r] & \prod_{w\in S_\psi\cup S_p} \frac{H^1(D_w,A)}{\mathscr{L}_w} \arrow[r] & H^1_{\mathscr{L}^\perp}(A^D)^* , \arrow[d] \\
& 0 & H^0(G,A^D)^* \arrow[l] & \prod_{w\in S_\psi\cup S_p} H^2(D_w,A) \arrow[l] & H^2(G,A) \arrow[l]
\end{tikzcd}$$

where $\mathscr{L}^\perp$ is the orthogonal complement of $\mathscr{L}$ in $\prod_{w\in S_p\cup S_\psi} H^1(G,A^D)$.

Proof. This lemma follows from Poitou-Tate duality (*cf.* Theorem 4.50 (4) [Hid00]). $\square$

Definition 2.5. Let $P \in \mathrm{ht}_1(\Lambda)$ and m be a positive integer. Let R be a quotient ring of Λ with $\pi_R : \Lambda \to R$ such that $J_R := \pi_R(P^m) \neq 0$. The Pontryagin dual $(R/J_R)^*$ of R/J_R is a discrete G-module on which G acts via $\pi_R \circ \Psi$. Define $(\Psi_1, \Sigma^1) = (\Psi, \Sigma^c)$ and $(\Psi_2, \Sigma^2) = (\Psi^D, \Sigma)$. For $\bullet = 1, 2$, we put $A_\bullet = \Psi_\bullet \otimes (R/J_R)^*$ and define

$$\begin{aligned}
\mathcal{S}^\bullet(R) &= \ker\left\{ H^1(G,A_\bullet) \to \prod_{w\in S_\psi} H^1(I_w,A_\bullet)^{D_w} \times \prod_{w\in\Sigma^\bullet} H^1(D_w,A_\bullet)\right\}.\\
T^1(R) &= H^0(G,A_2)\times \prod_{w\in\Sigma^1} H^0(D_w,A_1)\\
T^2(R) &= H^0(G,A_1)\times \prod_{w\in\Sigma^2} H^0(D_w,A_2)\\
H^1_{P,m}(R) &= \prod_{w\in S_\psi} H^1(I_w,A_1)^{D_w}.\\
H^2_{P,m}(R) &= \prod_{w\in S_\psi} H^0(D_w,A_2)\\
M^\bullet_{P,m}(R) &= \mathcal{S}^\bullet(R) \oplus T^\bullet(R).
\end{aligned}$$

By definition, $\mathcal{S}^1(\Lambda) = Sel^{\mathrm{str}}_{\mathcal{K}}(\Psi,\Sigma)$ and $\mathcal{S}^2(\Lambda) = Sel^{\mathrm{str}}_{\mathcal{K}}(\Psi^D,\Sigma^c)$.

2.2. **Proof of the theorem.** Notations are as in the previous subsection. We begin with the following key proposition.

Proposition 2.6. *Suppose that*

(1) $R \simeq \mathcal{O}[\![X_1,\cdots,X_n]\!]$ for some $n \geq 1$.

(2) $H^1_{P,m}(R)$ and $H^2_{P,m}(R)$ are pseudo-null R-modules.

Then we have

$$\mathrm{char}_R M^1_{P,m}(R) = \mathrm{char}_R M^2_{P,m}(R).$$

Proof. We will use the specialization principle developed by T. Ochiai in [Och05] and then proceed by induction on n. To simplify the notation we suppress the subscript and write $M^\bullet(R)$ (resp. $H^\bullet(R)$) for $M^\bullet_{P,m}(R)$ (resp. $H^\bullet_{P,m}(R)$), $\bullet = 1, 2$. We first assume $n = 1$ and $R \simeq \mathcal{O}[\![T]\!]$. By Theorem 2.3 (1) and (2) [Och08], it suffices to show

$$\ell_p(M^1(R)[f]) - \ell_p(M^2(R)[f]) \tag{2.2}$$

is uniformly bounded from above and below for all $f \in R$ prime to J_R. Set

$$\mathbf{H}^0(A_\bullet) = \prod_{w\in S_\psi} H^0(I_w,A_\bullet)\times \prod_{w\in\Sigma^\bullet} H^0(D_w,A_\bullet)$$

For $f \in R$ prime to J_R, we consider the following exact sequence.

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
H^1(G, A_\bullet)[f] & \xrightarrow{\gamma_1} & \prod_{w\in S_\psi} H^1(I_w, A_\bullet)[f] \times \prod_{w\in\Sigma^\bullet} H^1(D_w, A_\bullet)[f] \\
\uparrow & & \uparrow \\
H^1(G, A_\bullet[f]) & \xrightarrow{\gamma_2} & \prod_{w\in S_\psi} H^1(I_w, A_\bullet[f]) \times \prod_{w\in\Sigma^\bullet} H^1(D_w, A_\bullet[f]) \\
\uparrow & & \uparrow \\
H^0(G, A_\bullet)\otimes R/(f) & \xrightarrow{\gamma_3} & \mathbf{H}^0(A_\bullet)\otimes R/(f) \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

Then $\mathcal{S}^\bullet(R)[f] = \ker\gamma_1$ and $\mathcal{S}^\bullet(R/(f)) = \ker\gamma_2$. By the snake lemma, we have

$$\operatorname{Ker}\gamma_3 \to \mathcal{S}^\bullet(R/(f)) \to \mathcal{S}^\bullet(R)[f] \to \operatorname{Coker}\gamma_3.$$

In addition, since $\mathbf{H}^0(A_\bullet)$ and $H^0(G, A_\bullet)$ are annihilated by J_R, f is prime to the characteristic ideal of $H^0(G, A_\bullet)$ and $\mathbf{H}^0(A_\bullet)$. By Lemma 2.2 we deduce that $\ell_p(\operatorname{Ker}\gamma_3)$ and $\ell_p(\operatorname{Coker}\gamma_3)$ are uniformly bounded. Thus $\ell_p(\mathcal{S}^\bullet(R/(f))) - \ell_p(\mathcal{S}^\bullet(R)[f])$ is uniformly bounded. On the other hand, it is clear that $T^\bullet(R)[f] = T^\bullet(R/(f))$. Therefore the uniform boundedness of (2.2) is equivalent to showing

$$\ell_p(M^1(R/(f))) - \ell_p(M^2(R/(f))) \tag{2.3}$$

is uniformly bounded for all $f \in R$ prime to J_R.

We put $A = A_1[f] = \Psi \otimes (R/(f))^*$. Then $R/(f) = \mathcal{O}[\![T]\!]/(J_R, f)$ is a finite ring as f is prime to J_R. Then A is a finite $\mathbf{Z}_p$-module and the Cartier dual A^D of A is $A_2[f]$. Let $\mathscr{L} = \mathscr{L}(\Sigma)$ be the local condition defined in (2.1). Then it is well known that the orthogonal complement $\mathscr{L}^\perp$ of $\mathscr{L}$ is $\mathscr{L}(\Sigma^c)$. Note that $h^1_{\mathscr{L}}(A) = \ell_p(\mathcal{S}^1(R/(f)))$ and $h^1_{\mathscr{L}^\perp}(A^D) = \ell_p(\mathcal{S}^2(R/(f)))$. Because $D_w/I_w \simeq \hat{\mathbf{Z}}$ and A is a finite group, we have

$$\ell_p(H^0(D_w, A)) = \ell_p(H^1(D_w/I_w, A^{I_w})) \text{ for finite } w. \tag{2.4}$$

By Tate's formula of local and global Euler characteristics (*cf.* Theorem 2.8 and Theorem 5.1 [Mil06]), we find that $\chi(G, A) = -[\mathcal{F}\colon \mathbf{Q}]\ell_p(A)$, $\chi(D_w, A) = -[\mathcal{K}_w\colon \mathbf{Q}_p]\ell_p(A)$ for $w \in S_p$ ($\mathcal{K}$ is a CM field) and $\chi(D_w, A) = 0$ for $w \in S_\psi$. It follows from the ordinary assumption (Ord) that $[\mathcal{F} : \mathbf{Q}] = \sum_{w\in\Sigma^c}[\mathcal{K}_w\colon \mathbf{Q}_p]$; hence

$$\chi(G, A) = \sum_{w\in\Sigma^c\cup S_\psi} \chi(D_w, A). \tag{2.5}$$

We also have the equality $h^2_\Sigma(A) = h^0_\Sigma(A^D)$ by Tate local duality (*cf.* Corollary 2.3 *op.cit.*). By Lemma 2.4, (2.4) and (2.5) we find that

$$\begin{aligned}
h^1_{\mathscr{L}}(A) - h^1_{\mathscr{L}^\perp}(A^D) &= -\chi(G, A) + h^0(A) - h^0(A^D) + \sum_{w\in\Sigma^c\cup S_\psi} \chi(D_w, A) - h^0_{\Sigma^c}(A) + h^2_\Sigma(A) \\
&= h^0(A) - h^0(A^D) - h^0_{\Sigma^c}(A) + h^0_\Sigma(A^D).
\end{aligned}$$

We thus find that

$$\ell_p(M^1(R/(f))) - \ell_p(M^2(R/(f))) = h^1_{\mathscr{L}}(A) + h^0_{\Sigma^c}(A) + h^0(A^D) - (h^1_{\mathscr{L}^\perp}(A^D) + h^0_\Sigma(A^D) + h^0(A)) = 0.$$

This completes the proof when $n = 1$.

If $n \geq 2$, then by [Och08, Lemma 2.5] there exists a pseudo-null R-module N such that for any linear element $l \in \mathcal{L}^{(n)}_{\mathcal{O}}(M^1(R)) \cap \mathcal{L}^{(n)}_{\mathcal{O}}(M^2(R)) \cap \mathcal{L}^{(n)}_{\mathcal{O}}(N) \cap \mathcal{L}^{(n)}_{\mathcal{O}}(H^1(R) \oplus H^2(R))$ (for the definitions of linear element and

$\mathcal{L}_{\mathcal{O}}^{(n)}$, see [Och08, Definition 2.2]) and l is prime to J_R, the kernels and cokernels of the natural $R/(l)$-module morphisms

$$H^1(G, A_\bullet[l]) \to H^1(G, A_\bullet)[l]$$
$$H^1(D_w, A_\bullet[l]) \to H^1(D_w, A_\bullet)[l],\ w \in S_p$$
$$H^1(I_w, A_\bullet[l]) \to H^1(I_w, A_\bullet)[l],\ w \in S_\psi$$

are pseudo-null $R/(l)$-modules, $\bullet = 1, 2$. Therefore we can deduce that

$$\mathrm{char}_{R/(l)} M^\bullet(R/(l)) = \mathrm{char}_{R/(l)} M^\bullet(R)[l].$$

Since $R/(l)$ is a $(n-1)$-variable Iwasawa algebra. By the choice of l, $H^1(R/(l))$ and $H^2(R/(l))$ are also pseudo-null $R/(l)$-modules. Therefore the assertion follows from the induction hypothesis and Prop.3.6 [Och05]. □

Lemma 2.7. *Let $w \notin S_p$. We have*

(1) $H^1(D_w/I_w, \Psi \otimes \Lambda^*) = 0$ for $w \notin S_\psi$.
(2) $H^1(I_w, \Psi \otimes \Lambda^*) = 0$ for $w \in S_\psi$.

Proof. We first prove (1). Since $w \notin S_\psi \cup S_p$, Ψ is unramified at w, and $\Psi_w(F_w) \neq 1$ because $\mathcal{K}_\infty$ contains the cyclotomic $\mathbf{Z}_p$-extension. Λ^* is divisible, and hence

$$H^1(D_w/I_w, \Psi \otimes \Lambda^*) = \Lambda^*/(\Psi(F_w) - 1)\Lambda^* = 0.$$

Next we prove (2). Let I_w^t be the maximal tame pro-p quotient of I_w and I' be the kernel of the quotient map $I_w \twoheadrightarrow I_w^t$. As w is prime to p, it is well known that I_w^t is also the maximal pro-p quotient of I_w and $I_w^t \simeq \mathbf{Z}_p(1)$. Let γ_t be a generator. Because $\Psi|_{I_w} \neq 1$ for $w \in S_\psi$ and Λ^* is divisible, we have

$$H^1(I_w, \Psi \otimes \Lambda^*) = H^1(I_w^t, (\Psi \otimes \Lambda^*)^{I'}) = (\Lambda^*)^{I'}/(\Psi(\gamma_t) - 1)(\Lambda^*)^{I'} = 0.$$

□

Now we are ready to prove our main result.

Theorem 2.8. *For every $P \in \mathrm{ht}_1(\Lambda)$, we have*

$$\mathrm{char}_\Lambda Sel_{\mathcal{K}}(\Psi, \Sigma) = \mathrm{char}_\Lambda Sel_{\mathcal{K}}(\Psi^D, \Sigma^c).$$

Proof. By definition, it is equivalent to

$$\ell_P(Sel_{\mathcal{K}}(\Psi, \Sigma)) = \ell_P(Sel_{\mathcal{K}}(\Psi^D, \Sigma^c))$$

for every $P \in \mathrm{ht}_1(\Lambda)$. Let $S = S_\psi$. Put $\mathscr{A} = \Psi \otimes \Lambda^*$. By Lemma 2.7 (2), we have the exact sequence

$$0 \to Sel_{\mathcal{K}}^{\mathrm{str}}(\Psi, \Sigma) \to Sel_{\mathcal{K}}(\Psi, \Sigma) \to \prod_{w \in \Sigma^c} H^1(D_w/I_w, \mathscr{A}^{I_w}). \tag{2.6}$$

Let $w \in S_p$. We claim that the image of D_w in $\mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$ has $\mathbf{Z}_p$-rank at least two. Indeed, if $\mathcal{F} = \mathbf{Q}$ and $\mathcal{K}$ is an imaginary quadratic field, this is clear, and if $d = [\mathcal{F} : \mathbf{Q}] > 1$, then the image of D_w in $\mathrm{Gal}(\mathcal{K}_\infty^-/\mathcal{K})$ already has $\mathbf{Z}_p$-rank greater than one [HT91, Lemma 4.2]). We let $\mathfrak{I}_w$ be the ideal generated by $\Psi(D_w) - 1$. Then $\mathfrak{I}_w$ has height greater than one, and it follows that $H^1(D_w/I_w, \mathscr{A}^{I_w})$, which is annihilated by $\mathfrak{I}_w$, is a pseudo-null Λ-module because it . Therefore by (2.6) we conclude that

$$Sel_{\mathcal{K}}^{\mathrm{str}}(\Psi, \Sigma) \sim Sel_{\mathcal{K}}(\Psi, \Sigma). \tag{2.7}$$

Note that (2.7) implies $Sel_{\mathcal{K}}^{\mathrm{str}}(\Psi, \Sigma)$ is a cotorsion Λ-module.

Now let m be a positive integer and let $\mathcal{S}^\bullet$ and $T^\bullet$ be as in Def. 2.5. For $w \in S_p$, the ideal $\mathfrak{I}_w$ also annihilates $\mathscr{A}^{D_w}$, so $\mathscr{A}^{D_w}$ and $T^\bullet(\Lambda)$ are pseudo-null Λ-modules. By Lemma 2.7 (2)

$$\mathcal{S}^\bullet(\Lambda) \sim Sel_{\mathcal{K}}^{\mathrm{str}}(\Psi^\bullet, \Sigma)[P^m],\ \bullet = 1, 2.$$

Note that for any discrete cotorsion Λ-module N, $\ell_P(N) = \ell_P(N[P^m])$ for $m >> 0$. We deduce that for $m >> 0$

$$\ell_P(Sel_{\mathcal{K}}(\Psi^\bullet, \Sigma)) = \ell_P(Sel_{\mathcal{K}}^{\mathrm{str}}(\Psi^\bullet, \Sigma)) = \ell_P(\mathcal{S}^\bullet(\Lambda)) = \ell_P(M_{P,m}^\bullet(\Lambda)).$$

If $w \in S_\psi$, then $\Psi(F_w) - 1 \notin p\Lambda$ and the ideal generated by $\Psi(I_w) - 1$ equals $(\Psi(I_w) - 1) = (\psi(I_w) - 1) = (p^r)$ for some integer r. Then $H^1(I_w, \mathscr{A}[P^m])^{D_w}$ and $H^0(D_w, \mathscr{A}^D[P^m])$ are annihilated by (p^r) and $\Psi(F_w) - 1$. It follows that

$$H^1_{P,m}(\Lambda) = \prod_{w \in S_\psi} H^1(I_w, \mathscr{A}[P^m])^{D_w} \text{ and } H^2_{P,m}(\Lambda) = \prod_{w \in S_\psi} H^0(D_w, \mathscr{A}[P^m])$$

are pseudo-null Λ-modules. The theorem follows from Prop. 2.6 directly. □

2.3. **Proof of the corollary.** Now we prove Cor. 1.2. By Theorem 1.1, it is equivalent to the following proposition.

Proposition 2.9. *If $\psi|_\Delta \neq 1$, we have*

$$\ell_P(Sel^S_{\mathcal{K}}(\Psi^D, \Sigma^c)) = \ell_P(Sel_{\mathcal{K}}(\Psi^D, \Sigma^c)) + \sum_{w \in S \setminus S_\psi} \mathrm{ord}_P((1 - \Psi(F_w)))$$

for every $P \in \mathrm{ht}_1(\Lambda)$.

Note that we have the following exact sequence by Lemma 2.7.

$$0 \to Sel_{\mathcal{K}}(\Psi^D, \Sigma^c) \to Sel^S_{\mathcal{K}}(\Psi^D, \Sigma^c) \xrightarrow{\gamma} \prod_{w \in S \setminus S_\psi} H^1(D_w, \Psi^D \otimes \Lambda^*). \tag{2.8}$$

Lemma 2.10. *Let $w \notin S_\psi \cup S_p$ be a finite prime of $\mathcal{K}$. Then for every $P \in \mathrm{ht}_1(\Lambda)$, we have*

$$\ell_P(H^1(D_w, \Psi^D \otimes \Lambda^*)) = \mathrm{ord}_P(1 - \Psi(F_w)).$$

Therefore by (2.8) *we have*

$$\ell_P(Sel^S_{\mathcal{K}}(\Psi^D, \Sigma^c)) \leq \ell_P(Sel_{\mathcal{K}}(\Psi^D, \Sigma^c)) + \sum_{w \in S \setminus S_\psi} \mathrm{ord}_P(1 - \Psi(F_w)).$$

Proof. Let $\mathscr{A} = \Psi \otimes \Lambda^*$ and $\mathscr{B} = \Psi^D \otimes \Lambda^*$. As $w \notin S_p \cup S_\psi$, I_w acts on $\mathscr{A}$ and $\mathscr{B}$ trivially. For $n \geq 1$, we put $\mathscr{A}_n = \mathscr{A}[P^n]$ and $\mathscr{B}_n = \mathscr{B}[P^n]$. Then $\mathscr{A}_n$ and $\mathscr{B}_n$ are cofinitely generated and cotorsion Λ-modules. Because D_w/I_w is topologically generated by the Frobenius F_w, we have $H^i(D_w/I_w, \mathscr{B}_n) = H^i(\mathbf{Z}_p, \mathscr{B}_n) = 0$ for $i > 1$. By Hochschild-Serre exact sequence, we have

$$0 \longrightarrow H^1(D_w/I_w, \mathscr{B}_n) \longrightarrow H^1(D_w, \mathscr{B}_n) \longrightarrow H^1(I_w, \mathscr{B}_n)^{D_w/I_w} \longrightarrow 0. \tag{2.9}$$

Recall that I^t_w is the maximal tame pro-p quotient of I_w. As w is prime to p, we have the following isomorphisms as D_w/I_w-modules.

$$H^1(I_w, \mathscr{B}_n) = H^1(I^t_w, \mathscr{B}_n) = \mathrm{Hom}(\mathbf{Z}_p(1), \mathscr{B}_n) \simeq \mathscr{A}_n. \tag{2.10}$$

In addition, $H^1(D_w/I_w, \mathscr{B}_n) = \mathscr{B}_n/(F_w - 1)\mathscr{B}_n$, and hence

$$\ell_P(H^1(D_w/I_w, \mathscr{B}_n)) = \ell_P(H^0(D_w, \mathscr{B}_n)). \tag{2.11}$$

Put (2.9), (2.10) and (2.11) together, we obtain that

$$\ell_P(H^1(D_w, \mathscr{B}_n)) = \ell_P(H^0(D_w, \mathscr{B}_n)) + \ell_P(H^0(D_w, \mathscr{A}_n)). \tag{2.12}$$

By the equality $\ell_P(H^0(D_w, \mathscr{B}) \otimes \Lambda/P^n) = \ell_P(H^0(D_w, \mathscr{B}_n))$, (2.12) and the exact sequence

$$0 \longrightarrow H^0(D_w, \mathscr{B}) \otimes \Lambda/P^n \longrightarrow H^1(D_w, \mathscr{B}_n) \longrightarrow H^1(D_w, \mathscr{B})[P^n] \longrightarrow 0,$$

we deduce that for all $n \geq 1$

$$\ell_P(H^1(D_w, \mathscr{B})[P^n]) = \ell_P(H^0(D_w, \mathscr{A}_n)) = \ell_P(H^0(D_w, \mathscr{A})[P^n]). \tag{2.13}$$

Since $\mathcal{K}_\infty$ contains the cyclotomic $\mathbf{Z}_p$-extension, $\Psi|_{D_w} \neq 1, \omega$. It follows that $H^1(D_w, \mathscr{B})$ and $H^0(D_w, \mathscr{A})$ are cotorsion. By (2.13), we have

$$\ell_P(H^1(D_w, \mathscr{B})) = \ell_P(H^1(D_w, \mathscr{B})[P^\infty]) = \ell_P(H^0(D_w, \mathscr{A})).$$

By the above equality we can deduce the lemma from the following the equality of Λ-modules.

$$H^0(D_w, \mathscr{A})^* = H^0(D_w, \Psi \otimes \Lambda^*)^* = \Lambda/(\Psi(F_w) - 1)\Lambda.$$

□

In virtue of Lemma 2.10 and the exact sequence (2.8), to prove Prop. 2.9, it suffices to prove the cokernel of the map γ in (2.8) is a pseudo-null Λ-module. This follows from the following stronger proposition due to [GV00].

Proposition 2.11. *If $\psi|_\Delta \neq 1$, the restriction map*

$$H^1(G_S, \Psi^D \otimes \Lambda^*) \to \prod_{w \in S} H^1(D_w, \Psi^D \otimes \Lambda^*) \times \prod_{w \in \Sigma^c} H^1(I_w, \Psi^D \otimes \Lambda^*).$$

has finite cokernel.

Proof. Since $Sel^{\mathrm{str}}_{\mathcal{K}}(\Psi^D, \Sigma^c)$ is a cotorsion Λ-module and $H^0(\mathcal{K}_\infty, \Psi \otimes \Lambda^*) = 0$ if $\psi|_\Delta \neq 1$, we can proceed the proof as in [GV00, Prop. 2.1] (because $\mathcal{K}_\infty$ contains the cyclotomic $\mathbf{Z}_p$-extension). □

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