THE DERIVATIVE FORMULA OF p-ADIC L-FUNCTIONS FOR IMAGINARY QUADRATIC FIELDS AT TRIVIAL ZEROS

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To Bernadette Perrin-Riou, on her 65th birthday

Abstract. The rank one Gross conjecture for Deligne-Ribet p-adic L-functions was solved in [DDP11] and [Ven15] by the Eisenstein congruence among Hilbert modular forms. The purpose of this paper is to prove an analogue of the Gross conjecture for the Katz p-adic L-functions attached to imaginary quadratic fields via the congruences between CM forms and non-CM forms. The new ingredient is to apply the p-adic Rankin-Selberg method to construct a non-CM Hida family which is congruent to a Hida family of CM forms at the $1+\varepsilon$ specialization.

Résumé. Le conjecture de Gross en rang 1 pour les fonctions L p-adiques de Deligne-Ribet a été résolue par [DDP11] et [Ven15] au moyen de congruences d'Eisenstein parmi les formes modulaires de Hilbert. Le but de cet article est de prouver un analogue de la conjecture de Gross pour les fonctions L p-adiques de Katz des corps quadratiques imaginaires, via les congruences entre formes CM et formes non-CM. Le nouvel ingrédient est l'application de la méthode de Rankin-Selberg p-adique pour construire une famille de Hida non-CM qui est congruente à une famille de Hida de formes CM pour la spécialisation $1+\varepsilon$.

Contents

1. Introduction

In [DDP11], Darmon, Dasgupta and Pollack applied the congruence between Eisenstein series and cusp forms to prove the rank one Gross conjecture for Deligne-Ribet p-adic L-functions with some assumptions, which were later removed by [Ven15]. The purpose of this paper is to apply their ideas in the setting of CM

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congruence to prove an analogue of Gross conjecture for the cyclotomic Katz p-adic L-functions associated with ring class characters of imaginary quadratic fields. To begin with, we let K be an imaginary quadratic field and let $p > 2$ be a rational prime. Fixing an isomorphism $\iota_p : \mathbf{C} \simeq \overline{\mathbf{Q}}_p$ once and for all, let $\mathfrak{p}$ be the prime above p induced by ι_p. We shall assume that

$$p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}, \quad \mathfrak{p} \neq \overline{\mathfrak{p}}.$$

Let $\mathfrak{f}$ be a prime-to-p ideal of $\mathcal{O}_K$. Let $K(\mathfrak{f})$ and $K(p^\infty)$ be the ray class fields of K of conductor $\mathfrak{f}$ and p^∞. To any p-adic character $\widehat{\phi} : \mathrm{Gal}(K(p^\infty)/K) \to \overline{\mathbf{Q}}_p^\times$ which is Hodge-Tate, one can associate a character $\phi : \mathrm{W}_K \to \mathbf{C}^\times$ of the Weil group W_K of K unramified outside places above p with $\widehat{\phi}(\mathrm{Fr}_\mathfrak{q}) = \iota_p(\phi(\mathrm{Fr}_\mathfrak{q}))$ for $\mathfrak{q} \nmid p$, where $\mathrm{Fr}_\mathfrak{q}$ denotes a geometric Frobenius at a prime $\mathfrak{q}$. The character $\widehat{\phi}$ is the p-adic avatar of ϕ (*cf.* [HT93, page 190]). Let $\mathcal{W}$ be a finite extension of the Witt ring $W(\overline{\mathbb{F}_p})$. Let $\chi : \mathrm{Gal}(K(\mathfrak{f})/K) \to \mathcal{W}^\times$ be a primitive ray class character modulo $\mathfrak{f}$. The works in [Kat78], [dS87] and [HT93] have proved the existence of a (two-variable) Katz p-adic L-function $\mathcal{L}_p(\chi)$ in the Iwasawa algebra $\mathcal{W}[\![\mathrm{Gal}(K(p^\infty)/K)]\!]$ characterized uniquely by the following interpolation property: there exists a pair $(\Omega_p, \Omega_\infty) \in \mathcal{W}^\times \times \mathbf{C}^\times$ such that for any p-adic character $\widehat{\phi} : \mathrm{Gal}(K(p^\infty)/K) \to \overline{\mathbf{Q}}_p^\times$ which is crystalline of Hodge-Tate weight $(-k-j, j)$ with either $k \geq 1$ and $j \geq 0$ or $k \leq 1$ and $k + j > 0$,

$$\frac{\widehat{\phi}(\mathcal{L}_p(\chi))}{\Omega_p^{k+2j}} = \frac{1}{2(\sqrt{-1})^{k+j}}(1 - \chi\phi(\mathrm{Fr}_{\overline{\mathfrak{p}}}))(1 - \chi\phi(\mathrm{Fr}_\mathfrak{p}^{-1})p^{-1}) \cdot \frac{L(0, \chi\phi)}{\Omega_\infty^{k+2j}}. \tag{1.1}$$

Here $L(s, \chi\phi)$ is the *complete* L-function of $\chi\phi$ (*cf.* [Tat79, §3]). Let K_∞^+ be the cyclotomic $\mathbf{Z}_p$-extension of K. Let $\boldsymbol{\varepsilon}_{\mathrm{cyc}} : \mathrm{Gal}(K(p^\infty)/K) \to \mathrm{Gal}(K_\infty^+/K) \to \mathbf{Z}_p^\times$ be the p-adic cyclotomic character. Define the cyclotomic p-adic L-function $L_p(-, \chi) : \mathbf{Z}_p \to \mathcal{W}$ by

$$L_p(s, \chi) := \boldsymbol{\varepsilon}_{\mathrm{cyc}}^s(\mathcal{L}_p(\chi)).$$

In the remainder of the introduction, we suppose that

$$\chi \neq \mathbf{1} \text{ and } \chi(\mathrm{Fr}_{\overline{\mathfrak{p}}}) = 1. \tag{1.2}$$

The assumption (1.2) implies that $L_p(0, \chi) = 0$ by the p-adic Kronecker formula, and in view of Gross' conjecture for Deligne-Ribet p-adic L-functions, it is tempting to expect the leading coefficient of the Taylor expansion of $L_p(s, \chi)$ at $s = 0$ to be connected with certain $\mathscr{L}$-invariant, or rather p-adic regulator, and special values of a L-function. Along this direction, the work [BS19] provides an affirmative answer in most cases. We would like to remark that the results of [BS19] indeed include more general CM fields assuming some major open conjectures in algebraic number theory. We recall the (cyclotomic) $\mathscr{L}$-invariant associated with χ introduced in [BS19, Remark 1.5 (ii)]. Let $H = K(\mathfrak{f})$ be the ray class field of conductor $\mathfrak{f}$ and $\mathcal{O}_{H,\overline{\mathfrak{p}}}^\times$ be the group of $\overline{\mathfrak{p}}$-units. Put

$$\mathcal{O}_{H,\overline{\mathfrak{p}}}^\times[\chi] := \left\{ u \in \mathcal{O}_{H,\overline{\mathfrak{p}}}^\times \otimes_\mathbf{Z} \overline{\mathbf{Q}}_p \mid (\sigma \otimes 1)u = (1 \otimes \chi(\sigma))u \text{ for all } \sigma \in \mathrm{Gal}(H/K) \right\}.$$

We have $\mathcal{O}_{H,\overline{\mathfrak{p}}}^\times[\chi] = \mathrm{H}^1_{\{\overline{\mathfrak{p}}\}}(K, \chi^{-1}(1))$ via Kummer map. The dimension of the space is given by $\dim_{\overline{\mathbf{Q}}_p} \mathcal{O}_{H,\overline{\mathfrak{p}}}^\times[\chi] = 2$. Let $\mathfrak{P}$ be the prime of $\mathcal{O}_H$ induced by ι_p. By Dirichlet's units Theorem, we can choose a basis $\left\{ \mathfrak{u}^\chi, \mathfrak{u}_{\overline{\mathfrak{p}}}^\chi \right\}$ with $\mathfrak{u}^\chi \in \mathcal{O}_H^\times \otimes_\mathbf{Z} \overline{\mathbf{Q}}_p$

and $\mathfrak{u}^{\chi}_{\overline{\mathfrak{p}}} \in \mathcal{O}^{\times}_{H,\overline{\mathfrak{P}}} \otimes \overline{\mathbf{Q}}_p$ with $\mathrm{ord}_{\overline{\mathfrak{P}}}(\mathfrak{u}^{\chi}_{\overline{\mathfrak{p}}}) \neq 0$. Let c denote the complex conjugation. Define logarithms $\log_{\mathfrak{p}}$, $\log_{\overline{\mathfrak{p}}} : H^{\times} \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}_p \to \overline{\mathbf{Q}}_p$ by

$$\log_{\mathfrak{p}}(x \otimes \alpha) = \log_p(\iota_p(x))\alpha; \quad \log_{\overline{\mathfrak{p}}}(x \otimes \alpha) := \log_{\mathfrak{p}}(\overline{x} \otimes \alpha).$$

Let V_χ be the kernel of $\log_{\mathfrak{p}} : \mathcal{O}^{\times}_{H,\overline{\mathfrak{p}}}[\chi] \to \overline{\mathbf{Q}}_p$. Then V_χ is a one dimensional space generated by

$$u := \mathfrak{u}^{\chi}_{\overline{\mathfrak{p}}} \cdot (1 \otimes \log_{\mathfrak{p}} \mathfrak{u}^{\chi}) - \mathfrak{u}^{\chi} \cdot (1 \otimes \log_{\mathfrak{p}} \mathfrak{u}^{\chi}_{\overline{\mathfrak{p}}}). \tag{1.3}$$

The $\mathscr{L}$-invariant $\mathscr{L}(\chi)$ is defined by

$$\mathscr{L}(\chi) := -\frac{\log_{\overline{\mathfrak{p}}}(u)}{\mathrm{ord}_{\overline{\mathfrak{p}}}(u)} = \frac{1}{\log_{\mathfrak{p}} \mathfrak{u}^{\chi} \cdot \mathrm{ord}_{\overline{\mathfrak{P}}}(\mathfrak{u}^{\chi}_{\overline{\mathfrak{p}}})} \cdot \det \begin{pmatrix} \log_{\mathfrak{p}} \mathfrak{u}^{\chi}_{\overline{\mathfrak{p}}} & \log_{\mathfrak{p}} \mathfrak{u}^{\chi} \\ \log_{\overline{\mathfrak{p}}} \mathfrak{u}^{\chi}_{\overline{\mathfrak{p}}} & \log_{\overline{\mathfrak{p}}} \mathfrak{u}^{\chi} \end{pmatrix}. \tag{1.4}$$

Note that $\mathscr{L}(\chi)$ is a Gross-style regulator for imaginary quadratic fields. The above definition does not depend on the choice of basis. Let $\varphi_{\mathfrak{f}}(\mathcal{O}_K)$ be the Robert's unit in $H^{\times} \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}_p$ introduced in [dS87, p.55, (17)] and put

$$\mathfrak{e}_\chi := \sum_{\sigma \in \mathrm{Gal}(H/K)} \sigma(\varphi_{\mathfrak{f}}(\mathcal{O}_K)) \otimes \chi^{-1}(\sigma) \in (\mathcal{O}^{\times}_H \otimes \overline{\mathbf{Q}}_p)[\chi]. \tag{1.5}$$

We have the following Gross conjecture in the setting of imaginary quadratic fields

Conjecture 1. *For all primitive ray class characters χ of K modulo $\mathfrak{f}$ satisfying* (1.2)*, we have*

$$\frac{L_p(s,\chi)}{s}\Big|_{s=0} = \frac{-1}{12 w_{\mathfrak{f}}} \left(1 - \frac{\chi(\mathrm{Fr}_{\mathfrak{p}}^{-1})}{p}\right) \log_{\mathfrak{p}} \mathfrak{e}_\chi \cdot \mathscr{L}(\chi).$$

Here $w_{\mathfrak{f}}$ is the number of units in $\mathcal{O}^{\times}_K$ congruent to 1 *modulo* $\mathfrak{f}$.

When p does not divide the class number of K, a proof of Conjecture 1 is given in [BS19, Theorem 1.8]. In this paper, we offer an entirely different proof of Conjecture 1 for ring class characters, removing the hypothesis on p-indivisibility of the class number.

Theorem A. *Let d_K be the fundamental discriminant of $\mathcal{O}_K$. Suppose that χ is a ring class character and that $(\mathfrak{f}, d_K) = 1$. Then Conjecture 1 holds.*

Regarding the non-vanishing of $\mathscr{L}$-invariants, we remark that it is shown in [BD21, Proposition 1.11] that either $\mathscr{L}(\chi)$ or $\mathscr{L}(\chi^{-1})$ is non-zero and that the $\mathscr{L}$-invariant $\mathscr{L}(\chi)$ is non-zero if the Four Exponentials Conjecture holds.

The proof in [BS19] requires the full arsenal of Iwasawa theory for imaginary quadratic fields and the existence of elliptic units. In the case of general CM fields, their method relies on the existence of Rubin-Stark units in ray class fields, which is one of the major open conjectures in algebraic number theory. In contrast, we adapt the ideas in [DDP11], replacing the Eisenstein congruence with the CM congruence for elliptic modular forms. This approach is inspired by a series of works of Hida and Tilouine [HT91], [HT93] and [HT94] on CM congruences and the anticyclotomic main conjecture for CM fields and a recent work [BD21]. This method is *units-free* and more amenable to general CM fields as in [DDP11] at least under the Leopoldt conjecture for totally real fields. The details for general CM fields will appear in a future work. We now give a sketch of the proof of Theorem 1.

Cohomological interpretation of the $\mathscr{L}$-invariant. The staring point is the observation $\dim_F \mathrm{H}^1(K,\chi) = 1$ by the global Poitou-Tate duality. Let $\kappa \neq 0 \in \mathrm{H}^1(K,\chi)$. Write $\mathrm{loc}_{\overline{\mathfrak{p}}}(\kappa) \in \mathrm{H}^1(K_{\overline{\mathfrak{p}}},\chi) = \mathrm{Hom}(G_{K_{\overline{\mathfrak{p}}}}, \overline{\mathbf{Q}}_p)$. Let $\kappa_{\mathrm{ur}} : G_{K_{\overline{\mathfrak{p}}}} \to E$ be the unique unramified homomorphism sending the geometric Frobenius $\mathrm{Fr}_{\overline{\mathfrak{p}}}$ to 1 and κ_{cyc} be the p-adic logarithm of the p-adic cyclotomic character. Then

$$\mathrm{loc}_{\overline{\mathfrak{p}}}(\kappa) = x \cdot \kappa_{\mathrm{ur}} + y \cdot \kappa_{\mathrm{cyc}}.$$

In Lemma 4.1, we show that $y \neq 0$ and

$$\mathscr{L}(\chi) = \frac{x}{y}. \tag{1.6}$$

Therefore, to prove Theorem 1 we need to construct a non-zero cohomology class κ whose x and y coordinates can be evaluated explicitly and related to the derivatives of the Katz p-adic L-functions.

$\mathscr{A}$-adic modular forms and construction of cohomology classes. The construction of κ relies on the idea in [HT94] of using the congruence between p-adic families of CM forms and non-CM forms to prove anticyclotomic main conjectures. Let $\mathscr{A}$ be the ring of rigid analytic functions on the unit disk $\left\{s \in \overline{\mathbf{Q}}_p \mid |s|_p \leq 1\right\}$. For an integer $k \geq 1$, a prime-to-p positive integer N and a Dirichlet character ξ modulo N, let $S_k(N,\chi)$ be the space of elliptic cusp forms of weight k, level N and character ξ. Denote by $\mathbf{S}(N,\xi)$ the space of ordinary $\mathscr{A}$-adic modular forms of tame level N and character ξ, consisting of q-expansion $\mathcal{F}(s)(q) \in \mathscr{A}[\![q]\!]$ such that $\mathcal{F}(k)(q)$ is the q-expansion of some p-ordinary elliptic cusp form of weight $k+1$, level Np for all but finitely many $k \equiv 0 \pmod{p-1}$. Since χ is assumed to be a ring class character, we can write $\chi = \phi^{1-c}$ for some ray class character ϕ of conductor $\mathfrak{c}$ prime to $d_K p$. Note that the choice of ϕ is not unique. Let $N = d_K \mathrm{N}\mathfrak{c}$ and $\xi := \tau_{K/\mathbf{Q}}\phi_+$, where $\tau_{K/\mathbf{Q}} : (\mathbf{Z}/d_K\mathbf{Z})^\times \to \mathbf{C}^\times$ is the quadratic character associated with $K/\mathbf{Q}$ and $\phi_+ : (\mathbf{Z}/\mathrm{N}\mathfrak{c}\mathbf{Z})^\times \to \mathbf{C}^\times$ is given by $\phi_+(a) = \phi(a\mathcal{O}_K)$. Let $\boldsymbol{\theta}_\phi$ and $\boldsymbol{\theta}_{\phi^c}$ be $\mathscr{A}$-adic CM forms in $\mathbf{S}(N, \phi_+\tau_{K/\mathbf{Q}})$ associated with ϕ and ϕ^c defined in (2.3). Let $\mathscr{K} = \mathrm{Frac}\,\mathscr{A}$. The theory of $\mathscr{A}$-adic newforms yields a decomposition of Hecke modules

$$\mathbf{S}(N, \phi_+\tau_{K/\mathbf{Q}}) = \mathscr{K}\boldsymbol{\theta}_\phi \oplus \mathscr{K}\boldsymbol{\theta}_{\phi^c} \oplus \mathbf{S}^\perp. \tag{1.7}$$

The submodule $\mathbf{S}^\perp$ interpolates the orthogonal complement of the space spanned by $\boldsymbol{\theta}_\phi$ and $\boldsymbol{\theta}_{\phi^c}$. Let $\mathbf{T}^\perp$ be the $\mathscr{A}$-algebra generated by the Hecke operators acting on $\mathbf{S}^\perp$. Suppose we are given a Hecke eigensystem $\lambda : \mathbf{T}^\perp \to \mathscr{A}^\dagger/(s^2)$ and a character $\Psi : G_K \to \mathscr{A}^\dagger/(s^2)$ such that

(a) $\Psi \equiv \phi \pmod{s}$,

(b) $\lambda(T_\ell) = \Psi(\mathrm{Fr}_{\mathfrak{l}}) + \Psi(\mathrm{Fr}_{\overline{\mathfrak{l}}})$ for $\ell = \mathfrak{l}\overline{\mathfrak{l}}$ split in K.

Write $\Psi = \phi(1+\psi' s) \pmod{s^2}$. In Theorem 4.2, we use the argument in [DDP11, §4] to construct a non-zero cohomology class $\kappa \in \mathrm{H}^1(K,\chi)$ such that

$$\mathrm{loc}_{\overline{\mathfrak{p}}}(\kappa) = \psi'|_{G_{K_{\overline{\mathfrak{p}}}}} - \phi(\overline{\mathfrak{p}})^{-1}\lambda(U_p)'(0) \cdot \kappa_{\mathrm{ur}}. \tag{1.8}$$

Here $\lambda(U_p)'(0)$ is the first derivative of the U_p-eigenvalue $\lambda(U_p)$ at $s = 0$.

Construction of Hecke eigenforms modulo s^2. The problem boils down to constructing a Hecke eigensystem $\lambda : \mathbf{T}^\perp \to \mathscr{A}^\dagger/(s^2)$ as above and computing the derivative of the U_p-eigenvalue $\lambda(U_p)$. This is the main bulk of this paper and is achieved by applying the p-adic Rankin-Selberg method. For any $C \mid N$, let $\mathscr{G}_C(s) \in \mathscr{A}[\![q]\!]$ be the q-expansion defined by

$$\mathscr{G}_C(s) = 1 + 2\zeta_p(1-s)^{-1} \sum_{n=1}^{\infty} \left(\sum_{d \mid n, p \nmid n} d^{-1} \langle d \rangle^s \right) q^{Cn},$$

where $\zeta_p(s)$ is the p-adic Riemann zeta function. For $k \geq 2$, $\mathscr{G}_C(k)$ is the q-expansion of an p-ordinary Eisenstein series of weight k and level $\Gamma_0(Cp)$. From the spectral decomposition of $e_{\mathrm{ord}}(\theta^\circ_\phi \mathscr{G}_C) \in \mathbf{S}(N, \phi_+\tau_{K/\mathbf{Q}})$ in (1.7), we find that there exist $\mathcal{C}(\phi, \phi^c)$ and $\mathcal{C}(\phi^c, \phi)$ in $\mathscr{K}$ such that

$$e_{\mathrm{ord}}(\theta^\circ_\phi \mathscr{G}_C) = \mathcal{C}(\phi, \phi^c)\boldsymbol{\theta}_\phi + \mathcal{C}(\phi^c, \phi)\boldsymbol{\theta}_{\phi^c} + \mathscr{H}$$

for some $\mathscr{A}$-adic form $\mathscr{H} \in \mathbf{S}^\perp$. According to [HT93, Theorem 8.1], the coefficients $\mathcal{C}(\phi, \phi^c)$ and $\mathcal{C}(\phi^c, \phi)$ are essentially a product of two-variable Katz p-adic L-functions $\mathcal{L}_p(s, t, \chi)$ (See (3.3) for the definition). By Hida's p-adic Rankin-Selberg method, we will prove in Proposition 3.7 the following precise identity

$$\mathcal{C}(\phi^c, \phi)(s) = \frac{2\mathcal{L}_p(s, 0, \mathbf{1})\mathcal{L}_p(s, 0, \chi)}{L(0, \tau_{K/\mathbf{Q}})\mathcal{L}_p(s, -s, \chi)\zeta_p(1-s)} \cdot \frac{\langle d_K \rangle^s}{(1 - \boldsymbol{\varepsilon}^s_{\mathfrak{p}}(\mathrm{Fr}_{\overline{\mathfrak{p}}}))^2} \tag{1.9}$$

for some good choices of ϕ and C. Following a similar calculation in [Ven15, §3], we will see in Theorem 4.4 that the $\mathscr{A}$-adic form $\mathscr{H}$ produces an explicit Hecke eigensystem $\lambda_{\mathscr{H}} : \mathbf{T}^\perp \to \mathscr{A}^\dagger/(s^2)$ with the properties (a) and (b) and use (1.9) to show that the first derivative of $\lambda_{\mathscr{H}}(U_p)$ is given by the derivatives of the Katz p-adic L-functions. Putting all ingredients together, we prove Theorem 1 in §4.3.

Finally, in §5 we compare the definition of $\mathscr{L}$-invariants in (1.4) and Benois' $\mathscr{L}$-invariant in the setting of imaginary quadratic fields. First, Perrin-Riou [PR95] formulated a general conjecture for special values of p-adic L-functions at all integer points except for the exceptional zero case. Using an idea of Greenberg [Gre94] in the ordinary case, Benois [Ben14] gave a general definition of $\mathscr{L}$-invariant using (φ, Γ)-modules and formulated a trivial zero conjecture including the non-critical case. We will confirm that our formula is compatible with his conjecture in §5.2.

Notation and convention. If F is a local or global field of characteristic zero, let $\mathcal{O}_F$ be the ring of integers of F. Let G_F denote the absolute Galois group of F and let $C_F := F^\times$ if F is local and C_F be the idele class group $\mathbf{A}_F^\times/F^\times$ if F is global. Let $\mathrm{rec}_F : C_F \to G_F^{ab}$ be the *geometrically normalized* reciprocity law homomorphism.

Let F be a global field. If $\mathfrak{q}$ is a prime ideal of $\mathcal{O}_F$ (resp. v is a place of F), let $F_\mathfrak{q}$ (resp. F_v) be the completion of F at $\mathfrak{q}$ (resp. v). Then $\mathrm{rec}_{F_\mathfrak{q}} : F_\mathfrak{q}^\times \to G_{F_\mathfrak{q}}^{ab}$ sends a uniformizer $\varpi_\mathfrak{q}$ of $\mathcal{O}_{F_\mathfrak{q}}$ to the corresponding geometric Frobenius $\mathrm{Fr}_\mathfrak{q}$. If S is a finite set of prime ideals of $\mathcal{O}_F$, let F_S be the maximal algebraic extension of F unramified outside S and let $G_{F,S} = \mathrm{Gal}(F_S/F)$. For a fractional ideal $\mathfrak{a}$ of a global field F, we let $\mathrm{Fr}_\mathfrak{a} := \prod_\mathfrak{q} \mathrm{Fr}_\mathfrak{q}^{n_\mathfrak{q}}$ if $\mathfrak{a}$ has the prime ideal factorization $\prod_\mathfrak{q} \mathfrak{q}^{n_\mathfrak{q}}$.

If $\chi : C_F \to \mathbf{C}^\times$ is an idele class character of $F^\times$ unramified outside S. If v is a place of F, let $\chi_v : F_v^\times \to \mathbf{C}^\times$ be the local component of χ at v and let $L(s, \chi_v)$ be the local L-factor of χ_v in [Tat79, (3.1)]. Let $L(s, \chi) = \prod_v L(s, \chi_v)$ is the *complete*

L-function of χ and $\epsilon(s,\chi)$ be the epsilon factor [Tat79, (3.5.1-2)]. If $\chi = \mathbf{1}$ is the trivial character, then we put $\zeta_{F_v}(s) = L(s, \mathbf{1}_v)$ and $\zeta_F(s) = L(s, \mathbf{1})$. In particular, $\zeta_{\mathbf{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ and $\zeta_{\mathbf{Q}}(2) = \pi/6$ under this definition. If χ is a character of $G_{F,S}$, we shall view χ as a Hecke character of C_F via rec_F and still denote by χ if there is no fear for confusion. Therefore,

$$\chi(\mathfrak{q}) := \chi(\mathrm{Fr}_{\mathfrak{q}}) = \chi_{\mathfrak{q}}(\varpi_{\mathfrak{q}}) \text{ for } \mathfrak{q} \notin S.$$

In particular, a primitive ray class character χ modulo $\mathfrak{c}$ shall be identified with an idele class character χ of F of conductor $\mathfrak{c}$.

We write $\mathbf{A} = \mathbf{A}_{\mathbf{Q}}$ for simplicity. Denote by $\mathbf{e} = \prod \mathbf{e}_v : \mathbf{A}/\mathbf{Q} \to \mathbf{C}^\times$ the unique additive character with $\mathbf{e}_\infty(x) = \exp(2\pi\sqrt{-1}x)$. If $\chi : \mathbf{A}^\times/\mathbf{Q}^\times \to \mathbf{C}^\times$ is a finite order idele class character of $\mathbf{Q}$ of level N, then let χ_{Dir} be the Dirichlet character modulo N obtained by the restriction of χ to $\prod_{\ell | N} \mathbf{Z}_\ell^\times$. With our convention, if $q \nmid N$ is a prime, then

$$\chi_q(q) = \chi((q)) = \chi_{\mathrm{Dir}}(q)^{-1}. \tag{1.10}$$

We fix an isomorphism $\iota_p : \mathbf{C} \simeq \overline{\mathbf{Q}}_p$ once and for all. Let $\boldsymbol{\omega} : \mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q}) \to \mathbf{C}^\times$ be Galois character such that $\iota_p \circ \boldsymbol{\omega}$ is the p-adic Teichmüller character. Identifying $\boldsymbol{\omega}$ with an idele class character of $\mathbf{Q}$, we have

$$\iota_p(\boldsymbol{\omega}_{\mathrm{Dir}}(a)) \equiv a \pmod{p}; \quad L(s, \boldsymbol{\omega}) = L(s, \boldsymbol{\omega}_{\mathrm{Dir}}^{-1}).$$

2. Ordinary Λ-adic CM forms

2.1. **Ordinary Λ-adic forms.** If N is a positive integer, let $\mathcal{S}_k(N,\chi)$ denote the space of elliptic cusp forms of level $\Gamma_1(N)$ and nebentypus χ_{Dir}^{-1}. If $f \in \mathcal{S}_k(N,\chi)$ is a Hecke eigenform, let $\varphi_f := \Phi(f)$ be the associated automorphic form. Let $\mathbf{Q}_\infty$ be the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$ and $\Gamma_{\mathbf{Q}} = \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$. Define the Iwasawa algebra $\Lambda := \mathcal{W}[\![\Gamma_{\mathbf{Q}}]\!]$ and write $\sigma \mapsto [\sigma]$ for the inclusion of group-like elements $\Gamma_{\mathbf{Q}} \to \Lambda^\times$. If $\nu : \Gamma_{\mathbf{Q}} \to \overline{\mathbf{Q}}_p^\times$ is a continuous character, we extend ν uniquely to a $\mathcal{W}$-algebra homomorphism $\nu : \Lambda \to \overline{\mathbf{Q}}_p^\times$ by the formula $\nu([\gamma]) = \nu(\gamma)$. Let $\boldsymbol{\varepsilon}_{\mathrm{cyc}} : \Gamma_{\mathbf{Q}} \to 1 + p\mathbf{Z}_p$ be the cyclotomic character. For $s \in \mathbf{Z}_p$, let P_s be the kernel of $\boldsymbol{\varepsilon}_{\mathrm{cyc}}^s : \Lambda \to \mathbf{Z}_p$, i.e. the ideal of Λ generated by $\left\{[\sigma] - \boldsymbol{\varepsilon}_{\mathrm{cyc}}^s(\sigma) \mid \sigma \in \Gamma_{\mathbf{Q}}\right\}$. For a positive prime-to-p integer N and a finite order idele class character χ modulo pN, let $\mathbf{S}^{\mathrm{ord}}(N,\chi,\Lambda)$ be the space of (ordinary) Λ-adic cusp forms of tame level N with nebentypus χ_{Dir}^{-1}, consisting of q-expansions $\mathcal{F}(q) = \sum_n \mathbf{a}(n,\mathcal{F})q^n \in \Lambda[\![q]\!]$ such that for $k \geq 1$, the specialization $\mathcal{F} \pmod{P_k} = \sum_n \boldsymbol{\varepsilon}_{\mathrm{cyc}}^k(\mathbf{a}(n,\mathcal{F}))q^n$ is the q-expansion of some cusp form $\mathcal{F}_k$ in $\mathcal{S}_{k+1}^{\mathrm{ord}}(pN, \chi\boldsymbol{\omega}^k) \otimes_{\mathbf{C},\iota_p} \overline{\mathbf{Q}}_p$ at the infinity cusp.

If R is a Λ-algebra which is an integral domain and finite over Λ, let $\mathbf{S}^{\mathrm{ord}}(N,\chi,R) := \mathbf{S}^{\mathrm{ord}}(N,\chi,\Lambda) \otimes_\Lambda R$ be the space of Λ-adic forms defined over R. A basic result in Hida theory asserts that $\mathbf{S}^{\mathrm{ord}}(N,\chi,R)$ is a free R-module of finite rank equipped with the action of Hecke operators $\{T_\ell\}_{\ell \nmid pN}, \{U_q\}_{q | pN}$. We let

$$\mathbf{T}(N,\chi,R) = R[\{T_\ell\}_{\ell \nmid pN}, \{U_q\}_{q|pN}] \subset \mathrm{End}_R\, \mathbf{S}^{\mathrm{ord}}(N,R,\chi)$$

be the big ordinary cuspidal Hecke algebra generated by these Hecke operators over R. By the freeness of $\mathbf{S}^{\mathrm{ord}}(N,\chi,R)$, we have $\mathbf{T}(N,\chi,R) = \mathbf{T}(N,\chi,\Lambda) \otimes_\Lambda R$. A prime ideal Q in $\mathrm{Spec}\, R$ is called an arithmetic point if Q is lying above P_k for some $k \geq 2$. A Λ-adic form $\mathcal{F}$ in $\mathbf{S}^{\mathrm{ord}}(N,\chi,R)$ is a *newform* of tame level $N_{\mathcal{F}} \mid N$ if for all but finite many arithmetic primes Q of $\operatorname{Spec} R$, the specialization $\mathcal{F} \pmod{Q} \in \mathcal{S}^{\mathrm{ord}}_{k+1}(pN_{\mathcal{F}}, \chi\boldsymbol{\omega}^k)$ is the q-expansion of a p-stabilized normalized elliptic newform of tame level $N_{\mathcal{F}}$.

2.2. **Classical CM forms.** Let K be an imaginary quadratic field and let $d_K > 0$ be the fundamental discriminant of $\mathcal{O}_K$. If ψ is an idele class character of K of conductor $\mathfrak{c}$ with $\psi_\infty(z) = z^{-k}$ for some non-negative integer k, we recall that the CM form associated with ψ is the elliptic modular form θ°_ψ of weight $k+1$ defined by the q-expansion

$$\theta^\circ_\psi = \sum_{(\mathfrak{a},\mathfrak{c})=1} \psi(\mathfrak{a}) q^{\mathrm{N}\mathfrak{a}},$$

where $\mathfrak{a}$ runs over ideals of $\mathcal{O}_K$ prime to $\mathfrak{c}$ and $\mathrm{N}\mathfrak{a} := \mathrm{N}_{K/\mathbf{Q}}(\mathfrak{a})$ is the norm of $\mathfrak{a}$. Write $\psi_+ := \psi|_{\mathbf{A}^\times_{\mathbf{Q}}} = |\cdot|^k_{\mathbf{A}_{\mathbf{Q}}}\omega$ for some finite order idele class character ω of $\mathbf{Q}$. Then θ°_ψ is a newform of weight $k+1$, level $\mathrm{N}\mathfrak{c}d_K$ and nebentypus $\omega^{-1}_{\mathrm{Dir}}$. Let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ lying above p. The $\mathfrak{p}$-stabilization $\theta^{(\mathfrak{p})}_\psi$ is defined by

$$\theta^{(\mathfrak{p})}_\psi = \sum_{(\mathfrak{a},\mathfrak{p}\mathfrak{c})=1} \psi(\mathfrak{a}) q^{\mathrm{N}\mathfrak{a}}.$$

2.3. **Λ-adic CM forms.** Suppose that $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$, where $\mathfrak{p}$ is the prime induced by the fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C} \simeq \mathbf{C}_p$. Let $K_{\mathfrak{p}^\infty}$ be the $\mathbf{Z}_p$-extension of K in $K(\mathfrak{p}^\infty)$ and $\Gamma_{K,\mathfrak{p}} = \mathrm{Gal}(K_{\mathfrak{p}^\infty}/K)$. Let $\mathfrak{c}$ be an ideal of $\mathcal{O}_K$ coprime to p. The transfer map $\mathscr{V} : G^{ab}_{\mathbf{Q}} \to G^{ab}_K$ induces a map $\mathscr{V} : \Gamma_{\mathbf{Q}} \to \mathrm{Gal}(K(p^\infty)/K) \to \Gamma_{K,\mathfrak{p}}$, which in turns gives rise to an embedding

$$\mathscr{V} : \Lambda = \mathcal{W}[\![\Gamma_{\mathbf{Q}}]\!] \to \Lambda_K := \mathcal{W}[\![\Gamma_{K,\mathfrak{p}}]\!] \tag{2.1}$$

such that $\mathscr{V}(\mathrm{rec}_{\mathbf{Q}_p}(z)|_{\mathbf{Q}_\infty}) = \mathrm{rec}_{K_\mathfrak{p}}(z)|_{K_{\mathfrak{p}^\infty}}$ for $z \in \mathbf{Q}^\times_p$. Let $\Psi^{\mathrm{univ}} : G_K \to \Lambda^\times_K$ be the universal character defined by the inclusion of group-like elements $\Gamma_{K,\mathfrak{p}} \to \Lambda^\times_K$

$$\Psi^{\mathrm{univ}}(\sigma) = [\sigma^{-1}|_{K_{\mathfrak{p}^\infty}}] \in \Lambda_K. \tag{2.2}$$

For any primitive ray class character ϕ modulo $\mathfrak{c}$, we define

$$\boldsymbol{\theta}_\phi(q) = \sum_{(\mathfrak{a},\mathfrak{p}\mathfrak{c})=1} \phi(\mathfrak{a}) \cdot \Psi^{\mathrm{univ}}(\mathrm{Fr}_\mathfrak{a}) q^{\mathrm{N}\mathfrak{a}} \in \Lambda_K[\![q]\!]. \tag{2.3}$$

Let $\phi_+ = \phi \circ \mathscr{V}$, regarded as an idele class character of $\mathbf{Q}$. Then $\boldsymbol{\theta}_\phi$ is a Λ-adic newform of tame level $N := d_K\mathrm{N}\mathfrak{c}$ and nebentypus $\phi_+\tau_{K/\mathbf{Q}}$, where $\tau_{K/\mathbf{Q}}$ is the quadratic character associated with $K/\mathbf{Q}$. Let $\mathbf{S} := \mathbf{S}^{\mathrm{ord}}(N, \phi_+\tau_{K/\mathbf{Q}}, \Lambda_K)$ and $\mathbf{T} := \mathbf{T}(N, \phi_+\tau_{K/\mathbf{Q}}, \Lambda_K)$. Then $\mathbf{S}$ is a free Λ_K-module with $\mathbf{T}$-action. Denote $\mathbf{K} = \mathrm{Frac}(\Lambda_K)$. Let $\mathbf{S}^\perp$ be the subspace of $\mathbf{S} \otimes \mathbf{K}$ generated by the following set

$$\Xi^\perp = \left\{ \mathcal{F}(q^M) \mid \mathcal{F} \neq \boldsymbol{\theta}_\phi \text{ or } \boldsymbol{\theta}_{\phi^c} \text{ a newform in } \mathbf{S} \text{ of tame level } N_{\mathcal{F}} \text{ and } MN_{\mathcal{F}} \mid N \right\}.$$

By the theory of Λ-adic newforms [Wil88, Prop. 1.5.2], we have the decomposition of $\mathbf{T}$-modules

$$\mathbf{S} \otimes_{\Lambda_K} \mathbf{K} = \mathbf{K} \cdot \boldsymbol{\theta}_\phi \oplus \mathbf{K} \cdot \boldsymbol{\theta}_{\phi^c} \oplus \mathbf{S}^\perp. \tag{2.4}$$

3. The p-adic Rankin-Selberg convolutions

3.1. **A classical Eisenstein series.** We recall a general construction of Eisenstein series in the theory of automorphic forms. If ω is a finite order idele class character of $\mathbf{Q}$ and k is an integer, let $\mathcal{A}_k(\omega)$ denote the space of automorphic forms $\varphi : \mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ such that

$$\varphi(zg\kappa_\theta) = \omega(z)\varphi(g)e^{2\pi\sqrt{-1}k\theta}, \quad z \in \mathbf{A}^\times,\ \kappa_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}_2(\mathbf{R}).$$

Let $\mathcal{A}^0_k(\omega) \subset \mathcal{A}_k(\omega)$ be the subspace of cusp forms. For $a \in \mathbf{Z}_p^\times$, put $\langle a\rangle := a\cdot(\iota_p\circ \boldsymbol{\omega})(a)^{-1}$. For each place v, let $\boldsymbol{\omega}_v$ be the local component of $\boldsymbol{\omega}$ at v. Let $\mathcal{D}$ be the pair

$$\mathcal{D} = (k, C), \quad C \in \mathbf{Z}_{>0} \text{ and } p \nmid C.$$

Let $\mathcal{S}(\mathbf{A}^2)$ be the space of Bruhat-Schwartz functions on $\mathbf{A}^2$. Define $\Phi_{\mathcal{D}} = \Phi_{\mathcal{D},\infty} \otimes'_\ell \Phi_{\mathcal{D},\ell} \in \mathcal{S}(\mathbf{A}^2)$ by

— $\Phi_{\mathcal{D},\infty}(x, y) = 2^{-k}(x + \sqrt{-1}y)^k e^{-\pi(x^2+y^2)}$,
— $\Phi_{\mathcal{D},\ell}(x, y) = \mathbb{I}_{C\mathbf{Z}_\ell}(x)\mathbb{I}_{\mathbf{Z}_\ell}(y)$,
— $\Phi_{\mathcal{D},p}(x, y) = \boldsymbol{\omega}_p^{-k}(x)\mathbb{I}_{\mathbf{Z}_p^\times}(x)\mathbb{I}_{\mathbf{Z}_p}(y)$.

Recall that $f_{\mathcal{D},s} = \otimes_v f_{\mathcal{D},s,v}$, where $f_{\mathcal{D},s,v} = f_{\boldsymbol{\omega}_v^k,\mathbf{1},\Phi_{\mathcal{D},v},s} : \mathrm{GL}_2(\mathbf{Q}_v) \to \mathbf{C}$ is the Godement section associated with $\Phi_{\mathcal{D},v}$ defined by

$$f_{\mathcal{D},s,v}(g_v) = \boldsymbol{\omega}_v^k(\det g)\,|\det g_v|_v^{s+\frac{1}{2}} \int_{\mathbf{Q}_v^\times} \Phi_{\mathcal{D},v}((0, t_v)g_v)\boldsymbol{\omega}^k(t_v)\,|t_v|_v^{2s+1}\,\mathrm{d}^\times t_v$$

(*cf.* [CH20, (4.1)]). Let $B(\mathbf{Q})$ be the upper triangular matrices in $\mathrm{GL}_2(\mathbf{Q})$. Then the Eisenstein series $E_{\mathbf{A}}(-, f_{\mathcal{D},s}) : \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ is the series defined by

$$E_{\mathbf{A}}(g, f_{\mathcal{D},s}) = \sum_{\gamma\in B(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{Q})} f_{\mathcal{D},s}(\gamma g) \in \mathcal{A}_k(\mathbf{1})$$

(*cf.* [Bum97, (7.8), page 351]). The series $E_{\mathbf{A}}(g, f_{\mathcal{D},s})$ is absolutely convergent for $\mathrm{Re}\,(s) > 1/2$ and can be analytically continued to the whole complex plane except at $s = \pm\frac{1}{2}$. Suppose that $k \geq 2$. For $z = x + \sqrt{-1}y \in \mathfrak{H} = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$, put

$$E_k(C)(z) = y^{-\frac{k}{2}} E_{\mathbf{A}}(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, f_{\mathcal{D},s})|_{s=\frac{1-k}{2}}.$$

Then $E_k(C)(z)$ defines a classical Eisenstein series of weight k and level $\Gamma_0(pC)$.

Proposition 3.1. *The Fourier expansion of $E_k(C)$ is given by*

$$E_k(C) = \frac{\langle C\rangle^k}{2C}\zeta_p(1-k) + \sum_{n>0, C|n} \mathbf{a}(n, E_k(C))q^n,$$

where

$$\mathbf{a}(n, E_k(C)) = \sum_{C|d|n,\ p\nmid d} d^{k-1}\boldsymbol{\omega}_{\mathrm{Dir}}(d)^{-k}. \tag{3.1}$$

Proof. For each positive integer n, the Fourier coefficient $\mathbf{a}(n, E_k(C))$ is the product of local Whittaker functions

$$\mathbf{a}(n, E_k(C)) = n^{\frac{k}{2}} \prod_\ell W(\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}, f_{\mathcal{D},s,\ell})|_{s=\frac{1-k}{2}},$$

where $W(-, f_{\mathcal{D},s,\ell}) : \mathrm{GL}_2(\mathbf{Q}_\ell) \to \mathbf{C}$ is the local Whittaker function defined by

$$W(g, f_{\mathcal{D},s,\ell}) = \lim_{n\to\infty} \int_{\ell^{-n}\mathbf{Z}_\ell} f_{\mathcal{D},s,\ell}(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g)\mathbf{e}_\ell(-x)\mathrm{d}x_\ell,$$

and the Haar measure $\mathrm{d}x_\ell$ is normalized so that $\mathrm{vol}(\mathbf{Z}_\ell, \mathrm{d}x_\ell) = 1$ (*cf.* [CH20, Corollary 4.7] and [Bum97, (7.14)]). Hence we get (3.1) from the explicit formulae of these local Whittaker functions in [CH20, Lemma 4.6].

On the other hand, the constant term $\mathbf{a}(0, E_k(C))$ of $E_k(C)$ at the infinity cusp is given by

$$\mathbf{a}(0, E_k(C)) = f_{\mathcal{D},\frac{1-k}{2}}(1) + (Mf_{\mathcal{D},s})(1)\Big|_{s=\frac{1-k}{2}},$$

where $Mf_{\mathcal{D},s}(g)$ is obtained by the analytic continuation of the intertwining integral

$$Mf_{\mathcal{D},s}(g) = \int_{\mathbf{A}} f_{\mathcal{D},s}(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})\mathrm{d}x,\ g \in \mathrm{GL}_2(\mathbf{A})$$

(*cf.* [Bum97, (7.15)]). A direct computation shows that for $\mathrm{Re}\,(s) \gg 0$,

$$\begin{aligned} Mf_{\mathcal{D},s}(1) =& \prod_v \int_{\mathbf{Q}_v} f_{\mathcal{D},s,v}(\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix})\mathrm{d}x_v \\ =& \frac{C^{2s}\prod_{\ell|C}\boldsymbol{\omega}_\ell^k(C)}{2} \cdot \frac{L(2s, \boldsymbol{\omega}^k)}{L(2s, \boldsymbol{\omega}_p^k)} \\ =& \frac{C^{2s}\boldsymbol{\omega}((C))^k}{2} \cdot L(2s, \boldsymbol{\omega}^k) \cdot \begin{cases} (1 - p^{-2s}\boldsymbol{\omega}_p^k(p)) & \text{if } \boldsymbol{\omega}_p^k \text{ is unramified,} \\ 1 & \text{if } \boldsymbol{\omega}_p^k \text{ is ramified,} \end{cases} \end{aligned}$$

(*cf.* [Bum97, Proposition 2.6.3 and (7.27)]). Since $f_{\mathcal{D},s,p}(1) = 0$, we see that

$$\begin{aligned} \mathbf{a}(0, E_k(C)) =& Mf_{\mathcal{D},s}(1)|_{s=\frac{1-k}{2}} \\ =& \frac{\langle C\rangle^k}{2C}(1 - p^{k-1}\boldsymbol{\omega}_{\mathrm{Dir}}^{-k}(p))L(1-k, \boldsymbol{\omega}_{\mathrm{Dir}}^{-k}) = \frac{\langle C\rangle^k}{2C}\zeta_p(1-k). \end{aligned}$$

This finishes the computation of the Fourier expansion of $E_k(C)$. □

Remark 3.2. Let $E_k(z)$ be the standard classical Eisenstein series with the q-expansion

$$E_k = \frac{\zeta(1-k)}{2} + \sum_{n>0} \sigma_{k-1}(n)q^n.$$

Let $E_k^{(p)}(z) := E_k(z) - p^{k-1}E_k(pz)$ be the p-stabilization of E_k. From the inspection of Fourier expansions, we have

$$E_k(C)(z) = C^{-1}\langle C\rangle^k \cdot E_k^{(p)}(Cz).$$

The adelic construction of $E_k(C)$ will be used in the later computation of the adelic Rankin-Selberg convolution.

3.2. **A Λ-adic Eisenstein series.** Let P be the augmentation ideal of Λ. Let $\mathcal{L}_p^{\mathrm{KL}}(\mathbf{1}) \in P^{-1}\Lambda$ be the Kubota-Leopoldt p-adic L-function associated with trivial character, i.e. $\boldsymbol{\varepsilon}_{\mathrm{cyc}}^s(\mathcal{L}_p^{\mathrm{KL}}(\mathbf{1})) = \zeta_p(1-s)$. Define the q-expansion

$$\mathcal{E}_C := \frac{[\mathrm{Fr}_C]^{-1}}{2C} \cdot \mathcal{L}_p^{\mathrm{KL}}(\mathbf{1}) + \sum_{n>0,\, C|n} \mathbf{a}(n, \mathcal{E}_C) q^n,$$
$$\mathbf{a}(n, \mathcal{E}_C) = \sum_{C|d|n,\, p\nmid d} d^{-1}[\mathrm{Fr}_d]^{-1} \in \Lambda.$$

Proposition 3.3. *The q-expansion $\mathcal{E}_C$ defines a Λ-adic form of Eisenstein series. More precisely, for $k \geq 2$, we have*

$$\boldsymbol{\varepsilon}_{\mathrm{cyc}}^k(\mathcal{E}_C) = E_k(C)(q).$$

Proof. Note that with our convention (1.10), for any positive integer a prime to p, Fr_a is an element in $G_{\mathbf{Q}}$ corresponding to the ideal $(a) = a\mathbf{Z}$ and

$$\boldsymbol{\varepsilon}_{\mathrm{cyc}}(\mathrm{Fr}_a) = \langle a \rangle^{-1} = a^{-1}\boldsymbol{\omega}_{\mathrm{Dir}}(a).$$

The assertion thus follows from Proposition 3.1 immediately. $\square$

3.3. **Two-variable and improved Katz p-adic L-functions.** Let $\mathfrak{f}$ be an integral ideal of K. If χ is an idele class character of K with the conductor $\mathfrak{f}$. The (finite) Hecke L-function for χ is defined by the Dirichlet series

$$L_{\mathrm{fin}}(s, \chi) = \sum_{(\mathfrak{a}, \mathfrak{f})=1} \chi(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-s}.$$

If the infinity type of χ is $(a, b) \in \mathbf{Z}^2$, i.e. $\chi_\infty(z) = z^a \bar{z}^b$, then the Hecke L-function associated with χ is given by

$$L(s, \chi) := 2(2\pi)^{-(s+\max\{a,b\})}\Gamma(s + \max\{a, b\})L_{\mathrm{fin}}(s, \chi) \tag{3.2}$$

Suppose that $(\mathfrak{p}\bar{\mathfrak{p}}, \mathfrak{f}) = 1$. We consider the p-adic L-functions $L_{p,\mathfrak{f}\bar{\mathfrak{p}}^\infty}$ and $L_{p,\mathfrak{f}}$ of K defined in [dS87, (49), page 86]. Let χ be a primitive ray class character modulo $\mathfrak{f}$. Let $\mathcal{L}_p(\chi)$ be the unique element in the Iwasawa algebra $\mathcal{W}[\![\mathrm{Gal}(K(p^\infty)/K]\!]$ such that for every p-adic continuous character ϵ on $\mathrm{Gal}(K(p^\infty)/K)$, we have

$$\epsilon(\mathcal{L}_p(\chi)) = L_{p,\mathfrak{f}\bar{\mathfrak{p}}^\infty}(\chi\epsilon)\epsilon(\sigma_\delta),$$

where $\sigma_\delta \in \mathrm{Gal}(K(\mathfrak{f}p^\infty)/K(\mathfrak{f}\bar{\mathfrak{p}}^\infty))$ is the element defined in [dS87, (7), page 92]. We call $\mathcal{L}_p(\chi)$ the two-variable Katz p-adic L-function associated with χ. Let $\boldsymbol{\varepsilon}_{\mathfrak{p}} : \Gamma_{K,\mathfrak{p}} = \mathrm{Gal}(K_{\mathfrak{p}^\infty}/K) \to \mathcal{W}^\times$ be a p-adic character such that

$$\boldsymbol{\varepsilon}_{\mathfrak{p}}(\mathrm{rec}_K(z)) = \langle z_{\mathfrak{p}} \rangle,\ z \in \widehat{\mathcal{O}}_K^\times.$$

By definition, $\boldsymbol{\varepsilon}_{\mathfrak{p}} \circ \mathscr{V} = \boldsymbol{\varepsilon}_{\mathrm{cyc}}$. Let $\boldsymbol{\varepsilon}_{\bar{\mathfrak{p}}}(\sigma) := \boldsymbol{\varepsilon}_{\mathfrak{p}}(c\sigma c)$. It is convenient to introduce the two-variable Katz p-adic L-function $\mathcal{L}_p(s, t, \chi) : \mathbf{Z}_p^2 \to \mathcal{W}$ defined by

$$\mathcal{L}_p(s, t, \chi) := (\boldsymbol{\varepsilon}_{\mathfrak{p}}^s \boldsymbol{\varepsilon}_{\bar{\mathfrak{p}}}^t)(\mathcal{L}_p(\chi)) \text{ for } (s, t) \in \mathbf{Z}_p^2. \tag{3.3}$$

Let ψ be the idele class character of $K^\times$ such that $\widehat{\psi} = \boldsymbol{\varepsilon}_{\mathfrak{p}}$, i.e. $\psi : \mathbf{A}_K^\times/K^\times \to \mathbf{C}^\times$ is an idele class character of K unramified outside $\mathfrak{p}$ and $\psi_\infty(z) = z$ and $\psi(\mathfrak{q}) = \psi_{\mathfrak{q}}(\varpi_{\mathfrak{q}}) = \boldsymbol{\varepsilon}_{\mathfrak{p}}(\mathrm{Fr}_{\mathfrak{q}})$ for any prime $\mathfrak{q} \neq \mathfrak{p}$.

Proposition 3.4. *There exists periods $(\Omega_\infty, \Omega_p) \in \mathbf{C}^\times \times \mathcal{W}^\times$ such that for all $(k,j) \in \mathbf{Z}^2$ such that $k \geq 1$ and $j \geq 0$ or $k \leq 1$ and $k+j > 0$, we have*

$$\frac{\mathcal{L}_p(k+j,-j,\chi)}{\Omega_p^{k+2j}} = \frac{1}{2(\sqrt{-1})^{k+j}}(1-\chi\psi^{k+j(1-c)}(\overline{\mathfrak{p}}))(1-\chi\psi^{k+j(1-c)}(\mathfrak{p}^{-1})p^{-1})\frac{L(0,\chi\psi^{k+j(1-c)})}{\Omega_\infty^{k+2j}}.$$

Proof. Let $(\Omega, \Omega_p) \in \mathbf{C}^\times \times \mathcal{W}^\times$ be the periods introduced in [dS87, Theorem 4.14, page 80] and put $\Omega_\infty := (2\pi)^{-1}\Omega\sqrt{d_K}$. Write $\varepsilon = \boldsymbol{\varepsilon}_{\mathfrak{p}}^{k+j}\boldsymbol{\varepsilon}_{\overline{\mathfrak{p}}}^{-j}$. One deduces the desired interpolation formula of $\mathcal{L}_p(k+j,-j,\chi) = L_{p,\mathfrak{f}\overline{\mathfrak{p}}^\infty}(\chi\varepsilon)\varepsilon(\sigma_\delta)$ from [dS87, (50), page 86 and Lemma (i), page 92]. □

Likewise we define $\mathcal{L}_{\mathfrak{p}}^*(\chi)$ to be the unique element in $\mathcal{W}[\![\mathrm{Gal}(K(\mathfrak{p}^\infty)/K)]\!]$ such that

$$\epsilon(\mathcal{L}_{\mathfrak{p}}^*(\chi)) = L_{p,\mathfrak{f}}(\chi\epsilon)\epsilon(\sigma_\delta)$$

for any p-adic character ϵ on $\mathrm{Gal}(K(\mathfrak{p}^\infty)/K)$. Put

$$\mathcal{L}_{\mathfrak{p}}^*(s,\chi) := \boldsymbol{\varepsilon}_{\mathfrak{p}}^s(\mathcal{L}_{\mathfrak{p}}^*(\chi)).$$

Then $\mathcal{L}_{\mathfrak{p}}^*(\chi)$ is called the (one-variable) improved p-adic L-function associated with χ in the sense that

$$\mathcal{L}_p(s,0,\chi) = (1-\chi(\overline{\mathfrak{p}})\boldsymbol{\varepsilon}_{\mathfrak{p}}^s(\mathrm{Fr}_{\overline{\mathfrak{p}}}))\mathcal{L}_{\mathfrak{p}}^*(s,\chi). \tag{3.4}$$

If $\chi \neq \mathbf{1}$, then by the p-adic Kronecker limit formula [dS87, Theorem 5.2, page 88], we have

$$\mathcal{L}_{\mathfrak{p}}^*(0,\chi) = \frac{-1}{12w_{\mathfrak{f}}}\left(1-\frac{\chi(\mathfrak{p}^{-1})}{p}\right)\log_p \mathfrak{e}_\chi, \tag{3.5}$$

where $\mathfrak{e}_\chi$ is the Robert's unit in (1.5). It follows that $\mathcal{L}_{\mathfrak{p}}^*(0,\chi) \neq 0$ by the Brumer-Baker Theorem.

Remark 3.5. Recall that the cyclotomic p-adic L-function $L_p(s,\chi) := \boldsymbol{\varepsilon}_{\mathrm{cyc}}^s(\mathcal{L}_p(\chi))$. Let h be the class number of K. Since $\boldsymbol{\varepsilon}_{\mathfrak{p}}^h\boldsymbol{\varepsilon}_{\overline{\mathfrak{p}}}^h = \boldsymbol{\varepsilon}_{\mathrm{cyc}}^h$, we have

$$\mathcal{L}_p(ht,ht,\chi) = L_p(ht,\chi) \text{ for } t \in \mathbf{Z}_p.$$

3.4. **Rankin-Selberg convolution with CM forms.** Let χ be a ring class character unramified outside pd_K. There exists a ray class character ϕ such that

$$\chi = \phi^{1-c}$$

(*cf.* [Hid06, Lemma 5.31]). Replacing ϕ by $\phi \cdot \xi \circ \mathrm{N}_{K/\mathbf{Q}}$ for a suitable Dirichlet character ξ, we may further assume ϕ satisfies the following minimal condition

(min) the conductor of ϕ is minimal among Dirichlet twists.

Since $\chi = \phi^{1-c}$ is unramified outside pd_K, this in particular implies that the conductor $\mathfrak{c}$ of ϕ has a decomposition

$$\mathfrak{c} = \mathfrak{c}_{\mathrm{i}}\mathfrak{c}_{\mathrm{s}}, \quad (\mathfrak{c}, pd_K) = 1;\ (\overline{\mathfrak{c}_{\mathrm{s}}}, \mathfrak{c}_{\mathrm{s}}) = 1,$$

where $\mathfrak{c}_{\mathrm{i}}$ is only divisible by primes inert in K and $\mathfrak{c}_{\mathrm{s}}$ is only divisible by primes split in K. The level of the associated CM form θ_ϕ° is $N = d_K C_{\mathrm{i}}^2 C_{\mathrm{s}}$, where C_{i} and C_{s} are positive integers satisfying $(C_{\mathrm{i}}) = \mathfrak{c}_{\mathrm{i}} \cap \mathbf{Z}$ and $(C_{\mathrm{s}}) = \mathfrak{c}_{\mathrm{s}} \cap \mathbf{Z}$. Put

$$C = d_K C_{\mathrm{i}} C_{\mathrm{s}}.$$

With the transfer map $\mathscr{V}:\Lambda\to\Lambda_K$ in (2.1), we define

$$\mathcal{G}_C := \mathscr{V}\left(\frac{2C}{\mathcal{L}_p^{\mathrm{KL}}(\mathbf{1})}\cdot\mathcal{E}_C\right)\in\Lambda_P[\![q]\!], \tag{3.6}$$

where Λ_P is the localization of Λ_K at P. By construction and the fact that $\zeta_p(s)$ has a simple pole at $s=1$, we find that

$$\mathcal{G}_C \equiv [\mathrm{Fr}_C^{-1}] \equiv 1 \,(\mathrm{mod}\ P). \tag{3.7}$$

Let e_{ord} be Hida's ordinary projector on the space of Λ-adic forms. The spectral decomposition of

$$e_{\mathrm{ord}}(\theta_\phi^\circ\mathcal{G}_C)\in\mathbf{S}=\mathbf{S}^{\mathrm{ord}}(N,\phi_+\tau_{K/\mathbf{Q}},\Lambda_K)$$

according to (2.4) allows us to make the following definition.

Definition 3.6. Let $\mathcal{C}(\phi,\phi^c)$ and $\mathcal{C}(\phi^c,\phi)$ be the unique elements in $\mathbf{K}$ such that

$$\mathscr{H} := e_{\mathrm{ord}}(\theta_\phi^\circ\mathcal{G}_C)-\mathcal{C}(\phi,\phi^c)\cdot\boldsymbol{\theta}_\phi-\mathcal{C}(\phi^c,\phi)\cdot\boldsymbol{\theta}_{\phi^c}\in\mathbf{S}^\perp. \tag{3.8}$$

Let $\mathbf{c}$ be the positive integer such that $\mathbf{c}\mathcal{O}_K$ is the conductor of χ.

Proposition 3.7. *With the ray class character ϕ and the integer C as above, we have*

$$\boldsymbol{\varepsilon}_{\mathfrak{p}}^s(\mathcal{C}(\phi^c,\phi))=\frac{2\mathcal{L}_p(s,0,\mathbf{1})\mathcal{L}_p(s,0,\chi)}{L(0,\tau_{K/\mathbf{Q}})\mathcal{L}_p(s,-s,\chi)\zeta_p(1-s)}\cdot\frac{\langle d_K\mathbf{c}\rangle^s}{(1-\boldsymbol{\varepsilon}_{\mathfrak{p}}^s(\mathrm{Fr}_{\overline{\mathfrak{p}}}))^2}.$$

Proof. This can be proved by Hida's p-adic Rankin-Selberg method. We shall use the representation theoretic approach in [CH20]. We follow the notation in [CH20, Section 5, Section 6]. It suffices to show that for all but finitely many positive integer k with $k\equiv 0\,(\mathrm{mod}\ p-1)$,

$$\boldsymbol{\varepsilon}_{\mathfrak{p}}^k(\mathcal{C}(\phi^c,\phi))=\frac{2\mathcal{L}_p(k,0,\mathbf{1})\mathcal{L}_p(k,0,\chi)}{L(0,\tau_{K/\mathbf{Q}})\mathcal{L}_p(k,-k,\chi)\zeta_p(1-k)}\cdot\frac{(d_K\mathbf{c})^k}{(1-\psi^k(\mathrm{Fr}_{\overline{\mathfrak{p}}}))^2}. \tag{3.9}$$

Here recall that ψ is the idele class character of K corresponding to $\boldsymbol{\varepsilon}_{\mathfrak{p}}$ with $\psi_\infty(z)=z$. To evaluate $\boldsymbol{\varepsilon}_{\mathfrak{p}}^k(\mathcal{C}(\phi^c,\phi))$, we consider the spectral decomposition

$$\begin{aligned}&\frac{2C}{\zeta_p(1-k)}\cdot e_{\mathrm{ord}}(\theta_\phi^\circ E_k(C))\\ =&\mathcal{C}_k(\phi,\phi^c)\cdot\theta^{(\mathfrak{p})}_{\phi\psi^{-k}}+\mathcal{C}_k(\phi^c,\phi)\cdot\theta^{(\mathfrak{p})}_{\phi^c\psi^{-k}}+\mathscr{H}_k\in\mathcal{S}_{k+1}(Np,\phi_+^{-1}\tau_{K/\mathbf{Q}}),\end{aligned} \tag{3.10}$$

where $\mathscr{H}_k$ is orthogonal to the space spanned by $\theta_{\phi^{-1}\psi^{-k}},\theta^{(\mathfrak{p})}_{\phi^{-1}\psi^{-k}}$, $\theta_{\phi^{-c}\psi^{-k}}$ and $\theta^{(\mathfrak{p})}_{\phi^{-c}\psi^{-k}}$ under the Petersson inner product. Since $\boldsymbol{\varepsilon}_{\mathfrak{p}}^k(\boldsymbol{\theta}_\phi)=\theta^{(\mathfrak{p})}_{\phi\psi^{-k}}$ is a p-stabilized newform of weight $k+1$, the decomposition (3.10) is indeed obtained by the image of (3.8) under the map $\boldsymbol{\varepsilon}_{\mathfrak{p}}^k$, and hence

$$\boldsymbol{\varepsilon}_{\mathfrak{p}}^k(\mathcal{C}(\phi^c,\phi))=\iota_p^{-1}(\mathcal{C}_k(\phi^c,\phi)).$$

Now we use the adelic Rankin-Selberg method to compute the value $\mathcal{C}_k(\phi^c,\phi)$. Let $f^\circ=\theta_{\phi^{-1}\psi^{-k}}$ and $g^\circ=\theta_\phi^\circ$ be the newforms associated with Hecke characters $\phi^{-1}\psi^{-1}$ and ϕ. Let $\omega:=\phi_+^{-1}\tau_{K/\mathbf{Q}}^{-1}$ viewed as an idele class character of $\mathbf{Q}$. Let $\varphi_{f^\circ}:=\varPhi(f^\circ)\in\mathcal{A}_{k+1}(\omega)$ and $\varphi_{g^\circ}=\varPhi(g^\circ)\in\mathcal{A}_1(\omega^{-1})$ be the automorphic newforms corresponding to f° and g° via the map $\varPhi$ in [CH20, (2.4)]. Let π_1 and π_2 be the *unitary* cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ associated with φ_{f° and

φ_{g°. The π_1 and π_2 are the automorphic inductions of the idele class characters $\phi^{-1}\psi^{-k}|\cdot|_{\mathbf{A}_K}^{\frac{k}{2}}$ and ϕ, and the automorphic forms φ_{f° and φ_{g° are normalized new vectors in π_1 and π_2. In addition, we have the equality of automorphic L-functions and Dirichlet series of modular forms

$$\begin{aligned}L(s,\pi_1) =&\Gamma_{\mathbf{C}}(s+\frac{k}{2})D(s+\frac{k}{2},f^\circ) = L(s+\frac{k}{2},\phi^{-1}\psi^{-k});\\ L(s,\pi_2) =&\Gamma_{\mathbf{C}}(s)L(s,g^\circ) = L(s,\phi).\end{aligned}$$

Let $f=\theta^{(\mathfrak{p})}_{\phi^{-1}\psi^{-k}}$ be the $\mathfrak{p}$-stablized newform associated with f° and let $\breve{f}:=\theta^{(\mathfrak{p})}_{\phi^c\psi^{-k}}$ be the specialization of $\boldsymbol{\varepsilon}^s_{\mathfrak{p}}(\boldsymbol{\theta}_{\phi^c})$ at $s=k$. Then the automorphic representation generated by the associated automorphic forms $\varphi_{\breve{f}}$ is the contragredient representation $\pi_1^\vee=\pi_1\otimes\omega^{-1}$. Define the $\mathbf{C}$-linear pairing $\langle\,,\,\rangle:\mathcal{A}^0_{-k-1}(\omega)\times\mathcal{A}_{k+1}(\omega^{-1})\to\mathbf{C}$ by

$$\langle\varphi_1,\varphi_2\rangle=\int\limits_{\mathbf{A}_{\mathbf{Q}}^\times\mathrm{GL}_2(\mathbf{Q})\backslash\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})}\varphi_1(g)\varphi_2(g)\mathrm{d}^{\mathrm{t}}g.$$

Here $\mathrm{d}^{\mathrm{t}}g$ is the Tamagawa measure of $\mathrm{PGL}_2(\mathbf{A})$. By [CH20, Proposition 5.2], for $n\gg 0$ large enough, we have

$$\mathcal{C}_k(\phi^c,\phi)=\frac{\langle\rho(\mathcal{J}_\infty t_n)\varphi_f,\varphi_{g^\circ}\cdot E_{\mathbf{A}}(-,f_{\mathcal{D},s-1/2})\rangle|_{s=1-\frac{k}{2}}}{\langle\rho(\mathcal{J}_\infty t_n)\varphi_f,\varphi_{\breve{f}}\rangle}\cdot\frac{2C}{\zeta_p(1-k)},$$

where $\mathcal{J}_\infty=\begin{pmatrix}-1&0\\0&1\end{pmatrix}\in\mathrm{GL}_2(\mathbf{R})$ and $t_n=\begin{pmatrix}0&p^{-n}\\-p^n&0\end{pmatrix}\in\mathrm{GL}_2(\mathbf{Q}_p)$. In order to explain the calculation of $\mathcal{C}_k(\phi^c,\phi)$ by the adelic Rankin-Selberg method, we need to prepare some notation from the theory of automorphic representations. For any cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A})$, let $\mathcal{W}(\pi)$ denote the Whittaker model of π associated with the additive character $\mathbf{e}:\mathbf{A}/\mathbf{Q}\to\mathbf{C}^\times$. For each place v of $\mathbf{Q}$, let $\mathcal{W}_v(\pi)$ be the local component of $\mathcal{W}(\pi)$ at v. For $(W_1,W_2)\in\mathcal{W}_v(\pi_1)\times\mathcal{W}_v(\pi_2)$, let $\Psi(W_1,W_2,f_{\mathcal{D},s,v})$ be the local zeta integral defined in [CH20, (5.10)]. If v is finite, let $W_{\pi,v}\in\mathcal{W}_v(\pi)$ be the new Whittaker function with $W_{\pi,v}(1)=1$ and if $v=\infty$ and π_∞ is discrete series, let $W_{\pi,v}$ be the Whittaker of minimal $\mathrm{SO}(2)$-type with $W_{\pi,\infty}(1)=1$ (*cf.* [CH20, §2.6.4]). For $\varphi\in\mathcal{A}_0(\omega)$, the Whittaker function $W_\varphi:\mathrm{GL}_2(\mathbf{A})\to\mathbf{C}$ is defined by

$$W_\varphi(g)=\int_{\mathbf{A}/\mathbf{Q}}\varphi(\begin{pmatrix}1&x\\0&1\end{pmatrix}g)\mathbf{e}(-x)\mathrm{d}x.$$

In our setting, the Whittaker functions of $\varphi_f\in\pi_1$ and $\varphi_{g^\circ}\in\pi_2$ are given by

$$W_{\varphi_f}=W^{\mathrm{ord}}_{\pi_1,p}\prod_{v\neq p}W_{\pi_1,v};\quad W_{\varphi_{g^\circ}}=\prod_v W_{\pi_2,v},$$

where $W^{\mathrm{ord}}_{\pi_1,p}\in\mathcal{W}(\pi_{1,p})$ is the ordinary Whittaker function characterized by $W^{\mathrm{ord}}_{\pi_1,p}(\begin{pmatrix}a&0\\0&1\end{pmatrix})=\alpha_f(a)\,|a|^{\frac{1}{2}}_{\mathbf{Q}_p}\,\mathbb{I}_{\mathbf{Z}_p}(a)$, where $\alpha_f:\mathbf{Q}_p^\times\to\mathbf{C}^\times$ is the unramified character with $\alpha_f(p)=\phi^{-1}\psi^{-k}(\overline{\mathfrak{p}})p^{-\frac{k}{2}}$ (See [CH20, Definition 2.1]). Following [Jac72, Chapter V] (*cf.* [CH20, (5.11)]), we have the identity

$$\langle \rho(\mathcal{J}_\infty t_n)\varphi_f, \varphi_{g^\circ} \cdot E_{\mathbf{A}}(-, f_{\mathcal{D},s})\rangle = \int_{\mathrm{PGL}_2(\mathbf{Q})\backslash \mathrm{PGL}_2(\mathbf{A})} \varphi_f(g\mathcal{J}_\infty t_n)\varphi_{g^\circ}(g)E_{\mathbf{A}}(g, f_{\mathcal{D},s})\mathrm{d}^{\mathrm{t}}g$$

$$= \frac{1}{\zeta_{\mathbf{Q}}(2)}\Psi(W^{\mathrm{ord}}_{\pi_1,p}, W_{\pi_2,p}, f_{\mathcal{D},s,p})\Psi(\rho(\mathcal{J}_\infty)W_{\pi_1,\infty}, W_{\pi_2,\infty}, f_{\mathcal{D},s,\infty}) \prod_{v\neq p,\infty} \Psi(W_{\pi_1,v}, W_{\pi_2,v}, f_{\mathcal{D},s,v}).$$

By the calculation in [CH20, Proposition 5.3] with $k_1 = k_3 = k+1$ and $k_2 = 1$, we find that

$$\begin{aligned}(3.11) \qquad &\langle \rho(\mathcal{J}_\infty t_n)\varphi_f, \varphi_{g^\circ} \cdot E_{\mathbf{A}}(-, f_{\mathcal{D},s-1/2})\rangle|_{s=1-\frac{k}{2}} \\ &= \frac{L(s, \pi_1 \times \pi_2)}{\zeta_{\mathbf{Q}}(2)[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)]} \cdot \frac{(\sqrt{-1})^k}{2^{k+2}} \cdot \Psi_p(s) \prod_{\ell | N} \Psi^*_\ell(s)\Big|_{s=1-\frac{k}{2}},\end{aligned}$$

where $L(s, \pi_1 \times \pi_2)$ is the Rankin-Selberg L-function for $\pi_1 \times \pi_2$, $\Psi^*_\ell(s)$ and $\Psi_p(s)$ are local zeta integrals defined by

$$\Psi^*_\ell(s) = \frac{\zeta_{\mathbf{Q}_\ell}(1)}{\zeta_{\mathbf{Q}_\ell}(2)\,|N|_{\mathbf{Q}_\ell}} \frac{\Psi(W_{\pi_{1,\ell}}, W_{\pi_{2,\ell}}, f_{\Phi_{\mathcal{D},\ell},s-1/2})}{L(s, \pi_{1,\ell} \times \pi_{2,\ell})} \text{ if } \ell \neq p,$$

$$\Psi_p(s) = \frac{\Psi(\rho(t_n)W^{\mathrm{ord}}_{\pi_{1,p}}, W_{\pi_2,p}, f_{\Phi_{\mathcal{D},p},s-1/2})}{L(s, \pi_{1,p} \times \pi_{2,p})}.$$

Note that N is the conductor of π_1 and π_2. Let $\mathrm{supp}(N)$ be the set of prime divisors of N. In [CH20, §5.1, page 220], to (π_1, π_2), we associate a decomposition $\mathrm{supp}(N) = \Sigma_{(\mathrm{i})} \sqcup \Sigma_{(\mathrm{ii})} \sqcup \Sigma_{(\mathrm{iii})}$, and in our case, $\ell \in \Sigma_{(\mathrm{i})}$ if $\ell \mid d_K C_{\mathrm{s}}$, $\ell \in \Sigma_{(\mathrm{ii})}$ if $\ell \mid C_{\mathrm{i}}$ and $\Sigma_{(\mathrm{iii})} = \emptyset$. According to the computation of local zeta integrals $\Psi^*_\ell(s)$ in [CH20, Lemma 6.3, Lemma 6.5] at $\ell \mid N$, we have

$$\Psi^*_\ell(s) = 1 \text{ if } \ell \mid d_K C_{\mathrm{s}}; \quad \Psi^*_\ell(s) = |C_{\mathrm{i}}|^{-1}_{\mathbf{Q}_\ell}(1+\ell^{-1}) \text{ if } \ell \mid C_{\mathrm{i}}.$$

We compute the local zeta integral $\Psi_p(s)$ by a similar calculation in [CH20, Lemma 6.1]. Put $W_1 = W^{\mathrm{ord}}_{\pi_1,p}$ and $W_2 = W_{\pi_2,p}$. Then $\Psi(\rho(t_n)W^{\mathrm{ord}}_{\pi_{1,p}}, W_{\pi_2,p}, f_{\Phi_{\mathcal{D},p},s-1/2})$ equals

$$\begin{aligned}&\frac{\zeta_{\mathbf{Q}_p}(2)}{\zeta_{\mathbf{Q}_p}(1)} \int_{\mathbf{Q}_p^\times}\int_{\mathbf{Q}_p} W_1(\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & x\end{pmatrix} t_n)W_2(\begin{pmatrix} -y & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & x\end{pmatrix})\,|y|^{s-1}_{\mathbf{Q}_p} \\ &\times f_{\Phi_{\mathcal{D},p},s-\frac{1}{2}}(\begin{pmatrix} 0 & -1 \\ 1 & x\end{pmatrix})\mathrm{d}x\mathrm{d}^\times y \\ =&\frac{\zeta_{\mathbf{Q}_p}(2)}{\zeta_{\mathbf{Q}_p}(1)} \int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p^\times} W_1(\begin{pmatrix} yp^n & 0 \\ 0 & p^{-n}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ -p^{2n}x & 1\end{pmatrix})W_2(\begin{pmatrix} -y & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & x\end{pmatrix})\,|y|^{s-1}_{\mathbf{Q}_p} \\ &\times \mathbb{I}_{\mathbf{Z}_p}(x)\mathrm{d}^\times y\mathrm{d}x \\ =&\frac{\zeta_{\mathbf{Q}_p}(2)\alpha_f|\cdot|^{\frac{1}{2}}_{\mathbf{Q}_p}(p^{2n})\omega_p^{-1}(p^n)}{\zeta_{\mathbf{Q}_p}(1)} \int_{\mathbf{Q}_p^\times} W_2(\begin{pmatrix} -y & 0 \\ 0 & 1\end{pmatrix})\alpha_f|\cdot|^{s-\frac{1}{2}}_{\mathbf{Q}_p}(y)\mathrm{d}^\times y \\ =&\frac{\omega_p^{-1}\alpha_f^2|\cdot|_{\mathbf{Q}_p}(p^n)\zeta_{\mathbf{Q}_p}(2)}{\zeta_{\mathbf{Q}_p}(1)} \cdot L(s, \pi_{2,p} \otimes \alpha_f),\end{aligned}$$

so we obtain that

$$\Psi_p(s) = \frac{\omega_p^{-1}\alpha_f^2|\cdot|_{\mathbf{Q}_p}(p^n)\zeta_{\mathbf{Q}_p}(2)}{\zeta_{\mathbf{Q}_p}(1)} \cdot L(s, \pi_{1,p} \times \pi_{2,p})(1-\psi^{-k}(\mathfrak{p})p^{-s-\frac{k}{2}})(1-\phi^{c-1}\psi^{-k}(\mathfrak{p})p^{-s-\frac{k}{2}}).$$

From the above equations with the equality of L-functions

$$L(s, \pi_1 \times \pi_2) = L(s + k/2, \theta_\phi^\circ \otimes \theta_{\phi^{-1}\psi^{-k}}) = L(s + k/2, \psi^{-k})L(s + k/2, \phi^{c-1}\psi^{-k}),$$

we find that (3.11) equals

$$\langle \rho(\mathcal{J}_\infty t_n)\varphi_f, \varphi_{g^\circ} \cdot E_{\mathbf{A}}(-, f_{\mathcal{D},s-1/2})\rangle|_{s=1-\frac{k}{2}} = \frac{L(1, \psi^{-k})L(1, \phi^{c-1}\psi^{-k})}{\zeta_{\mathbf{Q}}(2)[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)]} \cdot \frac{(\sqrt{-1})^k}{2^{k+2}}$$
$$\times (1 - \psi^{-k}(\mathfrak{p})p^{-1})(1 - \phi^{c-1}\psi^{-k}(\mathfrak{p})p^{-1}) \cdot \frac{\omega_p^{-1}\alpha_f^2|\cdot|_{\mathbf{Q}_p}(p^n)\zeta_{\mathbf{Q}_p}(2)}{\zeta_{\mathbf{Q}_p}(1)} \cdot C_{\mathrm{i}} \prod_{q|C_{\mathrm{i}}}(1 + q^{-1}).$$

On the other hand, by [Hsi21, Lemma 3.6],

$$\langle \rho(\mathcal{J}_\infty t_n)\varphi_f, \varphi_{\breve{f}}\rangle = \frac{\|f^\circ\|^2_{\Gamma_0(N)}\mathcal{E}(f, \mathrm{Ad})}{\zeta_{\mathbf{Q}}(2)[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)]} \cdot \frac{\omega_p^{-1}\alpha_f^2|\cdot|_{\mathbf{Q}_p}(p^n)\zeta_{\mathbf{Q}_p}(2)}{\zeta_{\mathbf{Q}_p}(1)},$$

where $\mathcal{E}(f, \mathrm{Ad}) = (1 - \chi\psi^{(1-c)k}(\mathfrak{p}^{-1})p^{-1})(1 - \chi\psi^{(1-c)k}(\overline{\mathfrak{p}}))$. By the minimal condition (min) of ϕ, the level of the newform $f^\circ = \theta_\phi^\circ$ is minimal among its Dirichlet twists. By [HT93, Theorem 7.1],

$$\|f^\circ\|^2_{\Gamma_0(N)} = 2^{-(k+1)}L(1, \pi_1, \mathrm{Ad}) \cdot N \prod_{\ell|C_{\mathrm{i}}}(1 + \ell^{-1}) \quad (N = CC_{\mathrm{i}}).$$

Put $\psi_- = \psi^{1-c}$. From the above equations, we deduce that

(3.12)
$$\mathcal{C}_k(\phi^c, \phi) = \frac{(\sqrt{-1})^k L(1, \psi^{-k})L(1, \phi^{c-1}\psi^{-k})}{2L(1, \pi_1, \mathrm{Ad})} \cdot \frac{(1 - \psi^{-k}(\mathfrak{p})p^{-1})(1 - \phi^{c-1}\psi^{-k}(\mathfrak{p})p^{-1})}{(1 - \chi\psi_-^k(\mathfrak{p})^{-1}p^{-1})(1 - \chi\psi_-^k(\overline{\mathfrak{p}}))} \cdot \frac{2}{\zeta_p(1 - k)}.$$

By the functional equations of L-functions, one has

$$\begin{aligned} L(1, \psi^{-k})L(1, \phi^{c-1}\psi^{-k}) &= \varepsilon(1, \psi^{-k})\varepsilon(1, \chi^{-1}\psi^{-k})L(0, \psi^k)L(0, \chi\psi^k), \\ L(1, \pi_1, \mathrm{Ad}) &= L(1, \tau_{K/\mathbf{Q}})L(1, \phi^{1-c}\psi^{(1-c)k}) \\ &= \sqrt{d_K}^{-1}\varepsilon(1, \chi\psi_-^k) \cdot L(0, \tau_{K/\mathbf{Q}})L(0, \chi\psi_-^k). \end{aligned}$$

Since ψ^k is unramified everywhere and $\psi_-|_{\mathbf{A}_{\mathbf{Q}}^\times} = 1$, we have

$$\begin{aligned} &\varepsilon(1, \psi^{-k}) = \sqrt{d_K}^{-1}(\sqrt{-d_K})^k, \quad \varepsilon(1, \chi\psi^{-k}) = \varepsilon(1, \chi)(\sqrt{-d_K}\mathbf{c})^k, \\ &\varepsilon(1, \chi\psi_-^k) = \varepsilon(1, \chi). \end{aligned}$$

It follows that

$$\text{(3.13)} \quad \frac{(\sqrt{-1})^k L(1, \psi^{-k})L(1, \phi^{c-1}\psi^{-k})}{L(1, \pi_1, \mathrm{Ad})} = \frac{L(0, \psi^k)L(0, \chi\psi^k)}{(\sqrt{-1})^k L(0, \tau_{K/\mathbf{Q}})L(0, \chi\psi_-^k)} \cdot (d_K\mathbf{c})^k.$$

By the interpolation formulae of the Katz p-adic L-function in Proposition 3.4, we find that

(3.14)
$$\begin{aligned}\frac{\mathcal{L}_p(k,0,\chi)\mathcal{L}_p(k,0,\mathbf{1})}{\mathcal{L}_p(k,-k,\chi)} =& \frac{1}{2(\sqrt{-1})^k}\frac{L(0,\chi\psi^k)L(0,\psi^k)}{L(0,\chi\psi_-^k)}\\ &\times\frac{(1-\psi^k(\mathfrak{p}^{-1})p^{-1})(1-\chi\psi^k(\mathfrak{p}^{-1})p^{-1})(1-\psi^k(\overline{\mathfrak{p}}))^2}{(1-\chi\psi_-^k(\mathfrak{p}^{-1})p^{-1})(1-\chi\psi_-^k(\overline{\mathfrak{p}}))}.\end{aligned}$$

Combining (3.12), (3.13) and (3.14), we obtain (3.9). □

Recall that Λ_P is the localization of Λ_K at the augmentation ideal P. Let h be the class number of K and let $\varpi\in K^\times$ be a generator of $\mathfrak{p}^h$. Put

$$\mathscr{L}(\mathbf{1}) := -\frac{\log_p\varpi}{h} = \frac{\log_p\overline{\varpi}}{h} \neq 0.$$

Corollary 3.8. *We have $\mathcal{C}(\phi^c,\phi)^{-1}\in P\Lambda_P$. Let $B(s) := \boldsymbol{\varepsilon}_{\mathfrak{p}}^s(\mathcal{C}(\phi^c,\phi)^{-1})$. Then*

$$\frac{d}{ds}B(s)\Big|_{s=0} = \frac{\mathcal{L}_p'(s,-s,\chi)|_{s=0}}{\mathcal{L}_{\mathfrak{p}}^*(0,\chi)}.$$

Proof. By Proposition 3.7,

$$B(s) = \mathcal{L}_p(s,-s,\chi)\cdot\frac{L(0,\tau_{K/\mathbf{Q}})}{2\mathcal{L}_p(s,0,\mathbf{1})\mathcal{L}_{\mathfrak{p}}^*(s,\chi)}\cdot\frac{\zeta_p(1-s)(1-\boldsymbol{\varepsilon}_{\mathfrak{p}}^s(\mathrm{Fr}_{\overline{\mathfrak{p}}}))}{\langle d_K\mathbf{c}\rangle^s}. \tag{3.15}$$

By the residue formula of the p-adic zeta function,

$$\zeta_p(1-s)(1-\boldsymbol{\varepsilon}_{\mathfrak{p}}^s(\mathrm{Fr}_{\overline{\mathfrak{p}}}))|_{s=0} = (p^{-1}-1)\mathscr{L}(\mathbf{1})\neq 0.$$

On the other hand, from Katz's p-adic Kronecker limit formula [dS87, Theorem 5.2, page 88] and the fact that $L(0,\tau_{K/\mathbf{Q}}) = 2h/\#(\mathcal{O}_K^\times)$, we deduce that

$$\mathcal{L}_p(0,0,\mathbf{1}) = (1-p^{-1})\frac{\log_p\overline{\varpi}^{-1}}{\#(\mathcal{O}_K^\times)} = (p^{-1}-1)\mathscr{L}(\mathbf{1})\cdot 2^{-1}L(0,\tau_{K/\mathbf{Q}})\neq 0.$$

By (3.5) and the Brumer-Baker theorem, $\mathcal{L}_{\mathfrak{p}}^*(0,\chi)\neq 0$ and $B(0)=0$. We thus conclude from (3.15) that $\mathcal{C}(\phi^c,\phi)^{-1}\in P\Lambda_P$ and the desired formula of the derivative $B'(0)$. □

4. Galois cohomology classes and $\mathscr{L}$-invariants

4.1. **Cohomological interpretation of $\mathscr{L}$-invariants.** Let $F = \operatorname{Frac}\mathcal{W}$. As in the previous section, $\chi : \mathrm{Gal}(K(\mathbf{c})/K)\to F^\times$ is a non-trivial ring class character unramified outside pd_K with $\chi(\overline{\mathfrak{p}}) = 1$, and ϕ is a ray class character of conductor $\mathfrak{c}$ with $\chi = \phi^{1-c}$. For a finite set S of primes of $\mathcal{O}_K$, denote by $\mathrm{H}^1_S(K,\chi)$ the subspace of cohomology classes unramified outside S. By the global Poitu-Tate duality, it is known that $\mathrm{H}^1_\emptyset(K,\chi) = \mathrm{H}^1_{\{\overline{\mathfrak{p}}\}}(K,\chi) = \{0\}$ and $\dim_F\mathrm{H}^1(K,\chi) = \dim_F\mathrm{H}^1_{\{\mathfrak{p},\overline{\mathfrak{p}}\}}(K,\chi) = 1$ (*cf.* [BD21, Proposition 1.3]). Let $\mathrm{loc}_{\overline{\mathfrak{p}}} : \mathrm{H}^1(K,\chi)\to\mathrm{H}^1(K_{\overline{\mathfrak{p}}},\chi) = \mathrm{Hom}(G_{K_{\overline{\mathfrak{p}}}},F)$ be the localization at $\overline{\mathfrak{p}}$. With the embedding $\iota_p : K\hookrightarrow\mathbf{Q}_p$, we identity $K_p := K\otimes\mathbf{Q}_p\simeq\mathbf{Q}_p\oplus\mathbf{Q}_p$ by $\alpha\otimes x\mapsto(\iota_p(\alpha)x,\iota_p(\overline{\alpha})x)$. Let

$$\mathrm{rec}_{K_p} : K_p^\times = \mathbf{Q}_p^\times\oplus\mathbf{Q}_p^\times\to C_K\xrightarrow{\mathrm{rec}_K} G_K^{ab}$$

be the composition of the natural inclusion $K_p^\times \hookrightarrow C_K = K^\times\backslash\mathbf{A}_K^\times$ and the reciprocity law map rec_K. Therefore, for any $\kappa \in \mathrm{H}^1(K,\chi)$, we can identify $\mathrm{loc}_{\overline{\mathfrak{p}}}(\kappa) \in \mathrm{Hom}(G_{K_{\overline{\mathfrak{p}}}}, \overline{\mathbf{Q}}_p)$ with an element in $\mathrm{Hom}(\mathbf{Q}_p^\times, \overline{\mathbf{Q}}_p)$ by

$$\mathrm{loc}_{\overline{\mathfrak{p}}}(\kappa)(a) = \kappa(\mathrm{rec}_{K_p}(1,a)) \text{ for } a \in \mathbf{Q}_p^\times.$$

Lemma 4.1. *Let κ be a non-zero class in $\mathrm{H}^1(K,\chi)$ and write*

$$\mathrm{loc}_{\overline{\mathfrak{p}}}(\kappa) = x \cdot \mathrm{ord}_p + y \cdot \log_p.$$

Then $y \neq 0$, and

$$\mathscr{L}(\chi) = \frac{x}{y}.$$

Proof. First we note that $y \neq 0$ since $\mathrm{H}^1_{\{\overline{\mathfrak{p}}\}}(K,\chi) = 0$. Let $\mathrm{loc}_{\mathfrak{p}}(\kappa) = w \cdot \mathrm{ord}_p + z \cdot \log_p$. By the relation $\langle \mathrm{loc}_{\mathfrak{p}}(\kappa), \mathrm{loc}_{\mathfrak{p}}(x)\rangle + \langle \mathrm{loc}_{\overline{\mathfrak{p}}}(\kappa), \mathrm{loc}_{\overline{\mathfrak{p}}}(x)\rangle = 0$ for $x = \mathfrak{u}^\chi$ or $\mathfrak{u}^\chi_{\overline{\mathfrak{p}}}$, we obtain the equations

$$\begin{aligned} &z \cdot \log_{\mathfrak{p}} \mathfrak{u}^\chi + y \cdot \log_{\overline{\mathfrak{p}}}(\mathfrak{u}^\chi) = 0; \\ &z \cdot \log_{\mathfrak{p}} \mathfrak{u}^\chi_{\overline{\mathfrak{p}}} + x \cdot \mathrm{ord}_{\overline{\mathfrak{P}}}(\mathfrak{u}^\chi_{\overline{\mathfrak{p}}}) + y \cdot \log_{\overline{\mathfrak{p}}} \mathfrak{u}^\chi_{\overline{\mathfrak{p}}} = 0. \end{aligned}$$

The lemma now follows. ☐

4.2. **Construction of a cohomology classes.** Let S be the set of prime factors of $\mathfrak{p}\mathfrak{c}$. Let $G_{K,S} = \mathrm{Gal}(K_S/K)$, where K_S is the maximal algebraic extension of K unramified outside S. Let $\mathbf{T}^\perp \subset \mathrm{End}\,\mathbf{S}^\perp$ be the image of the Hecke algebra $\mathbf{T} = \mathbf{T}(N, \phi_+\tau_{K/\mathbf{Q}}, \Lambda_K)$ restricted to $\mathbf{S}^\perp$. Then $\mathbf{T}^\perp$ is a finite flat Λ_K-algebra. Fix a generator γ_0 of $\Gamma_{K,\mathfrak{p}}$. Then Λ_K can be identified with $\mathcal{W}[\![X]\!]$ and the augmentation ideal P of Λ_K is the principal ideal generated by $X = \gamma_0 - 1$. We use the argument in [DDP11, Theorem 4.2] to construct nonzero cohomology classes in the following

Theorem 4.2. *Let $\lambda : \mathbf{T}^\perp \to \Lambda_P/(X^{n+2})$ be a Λ_K-algebra homomorphism. Let $\boldsymbol{\alpha} : G_{K_{\overline{\mathfrak{p}}}} \to \Lambda_P^\times$ be the unique unramified character such that $\boldsymbol{\alpha}(\mathrm{Fr}_{\overline{\mathfrak{p}}}) = \lambda(U_p)$. Suppose that there exists a character $\widetilde{\Psi} : G_{K,S} \to \Lambda_P/(X^{n+2})$ such that*

(i) $\widetilde{\Psi} \equiv 1 \,(\mathrm{mod}\ X)$,

(ii) $\lambda(T_\ell) = \phi\widetilde{\Psi}(\mathrm{Fr}_{\mathfrak{l}}) + \phi\widetilde{\Psi}(\mathrm{Fr}_{\overline{\mathfrak{l}}})$ *for all $\ell \nmid p$ splits in K and*

(iii) $\widetilde{\Psi}|_{G_{K_{\overline{\mathfrak{p}}}}} \equiv \phi^{-1}\boldsymbol{\alpha} - \eta X^{n+1} \,(\mathrm{mod}\ X^{n+2})$ *for some non-zero homomorphism $\eta : G_{K_{\overline{\mathfrak{p}}}} \to \overline{\mathbf{Q}}_p$.*

Then there exists $\kappa \neq 0 \in \mathrm{H}^1(K,\chi)$ such that

$$\mathrm{loc}_{\overline{\mathfrak{p}}}(\kappa) = \left.\frac{\phi^{-1}\boldsymbol{\alpha} - \widetilde{\Psi}|_{G_{K_{\overline{\mathfrak{p}}}}}}{X^{n+1}}\right|_{X=0} = \eta.$$

Proof. Let $\Lambda^\dagger \supset \Lambda_P$ be the local ring of rigid analytic functions around $X = 0$, i.e.

$$\Lambda^\dagger = \left\{ \sum_{n=0}^\infty a_n X^n \in F[\![X]\!] \;\middle|\; \text{there exist } r > 0 \text{ such that } \lim_{n\to\infty} |a_n|\, r^n = 0 \right\}.$$

Let $\mathbf{T}^\dagger = \mathbf{T}^\perp \otimes_{\Lambda_K} \Lambda^\dagger$ be a finite $\Lambda^\dagger$-algebra, and hence a finite product of henselian local rings. Let I be the kernel of the map $\lambda : \mathbf{T}^\dagger \to \Lambda^\dagger/(X^{n+2})$. Let $\mathscr{T} : G_{\mathbf{Q},S} \to \mathbf{T}^\perp \to \mathbf{T}^\dagger$ be the pseudo character defined by $\mathscr{T}(\mathrm{Fr}_\ell) = T_\ell$. The assumption (ii) implies that

$$\mathscr{T}|_{G_{K,S}} \equiv \phi\widetilde{\Psi} + \phi^c\widetilde{\Psi}^c \,(\mathrm{mod}\ I).$$

Since $\phi \neq \phi^c$, applying the theory of residually multiplicity free pseudo characters [BC09, Theorem 1.4.4]) to $\mathscr{T}|_{G_{K,S}}$, we obtain a continuous representation $\rho_\lambda : G_{K,S} \to \mathrm{GL}_2(\operatorname{Frac} \mathbf{T}^\dagger)$ such that the image of $\rho_\lambda(\mathbf{T}^\dagger[G_{K,S}])$ is a generalized matrix algebra of the form

$$\rho_\lambda(\mathbf{T}^\dagger[G_{K,S}]) = \begin{pmatrix} \mathbf{T}^\dagger & \mathfrak{t}_{12} \\ \mathfrak{t}_{21} & \mathbf{T}^\dagger \end{pmatrix},$$

where $\mathfrak{t}_{ij}$ are fractional $\mathbf{T}^\dagger$-ideals in $\operatorname{Frac} \mathbf{T}^\dagger$ and $\mathfrak{t}_{12}\mathfrak{t}_{21} \subset I$. Writing

$$\rho_\lambda(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \text{ for } \sigma \in G_{K,S},$$

then we have

$$a(\sigma) \equiv \phi\widetilde{\Psi}(\sigma) \pmod{I}; \quad d(\sigma) \equiv \phi\widetilde{\Psi}(c\sigma c) \pmod{I}.$$

Note that $\mathbf{T}^\dagger/I \simeq \Lambda^\dagger/(X^{n+2})$ is a local ring. Let Q be the maximal ideal of $\mathbf{T}^\dagger$ containing I. Let $R = \mathbf{T}^\dagger_Q$ be the localization of $\mathbf{T}^\dagger$ at Q. Then R is a finite flat and reduced $\Lambda^\dagger$-algebra since N is the tame conductor of θ°_ϕ. Put $R_{ij} := \mathfrak{t}_{ij} \otimes_{\mathbf{T}^\dagger} R$. By [HT94, Theorem 6.12], there exists $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(\operatorname{Frac} R)$ such that

$$\begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}(\sigma) & * \\ 0 & * \end{pmatrix} \text{ for all } \sigma \in G_{K_{\overline{\mathfrak{p}}}},$$

and hence

$$C \cdot b(\sigma) = A \cdot (\boldsymbol{\alpha}(\sigma) - a(\sigma)) \text{ for } \sigma \in G_{K_{\overline{\mathfrak{p}}}}. \tag{4.1}$$

Note that R_{12} is a faithful R-module. Otherwise writing $R \otimes_{\Lambda^\dagger} \operatorname{Frac} \Lambda^\dagger = \oplus_i L_i$ as a product of fields, we would have $R_{12} \otimes_R L_i = 0$ for some i, which implies that there exists a Hida family $\mathcal{F}_1 = \Theta_{\phi_1}$ of CM forms in $\operatorname{Spec} \mathbf{T}^\perp$ for some ray class character $\phi_1 \neq \phi$ or ϕ^c whose specialization at some arithmetic point P' above P agree with $\theta_\phi^{(\mathfrak{p})}$, which in turns suggests that $\phi_1 + \phi_1^c = \phi + \phi^c$, and $\phi_1 = \phi$ or ϕ^c, a contradiction.

Define the function $\mathscr{K} : G_{K,S} \to R_{12}$ by $\mathscr{K}(\sigma) = b(\sigma)/d(\sigma)$. For any R-submodule $J \supset QR_{12}$ of R_{12}, the reduction of $\mathscr{K}$ modulo J

$$\overline{\mathscr{K}} := b/d \pmod{J} = \phi^{-c} b \pmod{J} : G_{K,S} \to R_{12}/J$$

is a continuous one-cocycle in $Z^1(G_{K,S}, \chi \otimes R_{12}/J)$. We claim that if the class $[\overline{\mathscr{K}}] \in \mathrm{H}^1(K, \chi \otimes R_{12}/J)$ represented by $\overline{\mathscr{K}}$ is zero, then $R_{12} = J$. We can write $b(\sigma) \pmod{J} = (\phi^c(\sigma) - \phi(\sigma))z$ for some $z \in R_{12}/J$. Consider the $\rho_\lambda(R[G_{K,S}])$-module $(R_{12}/J, R/Q)^{\mathrm{t}}$. Then the line $R(z,1)^{\mathrm{t}} \subset (R_{12}/J, R/Q)^{\mathrm{t}}$ is stable under the action of $\rho_\lambda(R[G_{K,S}])$. On the other hand, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \rho_\lambda(R[G_{K,S}])$, so we find that $(0,1)^{\mathrm{t}} \in R(z,1)^{\mathrm{t}}$. This implies $z = 0$ and $b(\sigma) \pmod{J}$ is zero. Since R_{12} is the R-module generated by $\{b(\sigma)\}_{\sigma \in G_{K,S}}$, we conclude $R_{12} = J$. In particular, this shows that $\mathscr{K} \pmod{QR_{12}}$ represents a non-zero class κ in $\mathrm{H}^1(K, \chi \otimes R_{12}/Q)$.

Let R'_{12} be the submodule of R_{12} generated by $\{b_\sigma\}_{\sigma \in G_{K_{\overline{\mathfrak{p}}}}}$ and let $J := QR_{12} + R'_{12}$. Then $\overline{\mathscr{K}} : G_K \to R_{12}/J$ is a cocycle which is trivial at $\overline{\mathfrak{p}}$, and $[\overline{\mathscr{K}}] \in \mathrm{H}^1_{\{\overline{\mathfrak{p}}\}}(K, \chi) = \{0\}$. By the above claim, we find that $J = R_{12}$ and hence $R'_{12} = R_{12}$

by Nakayama's lemma. Then the assumption (iii) and (4.1) suggest that $C \in (\mathrm{Frac}\, R)^\times$ and

$$R_{12} = \frac{A}{C} X^{n+1} R.$$

Since $\chi(\mathfrak{p}) = 1$, $\phi = \phi^c$ on $G_{K_{\overline{\mathfrak{p}}}}$, and it follows that for $\sigma \in G_{K_{\overline{\mathfrak{p}}}}$,

$$\mathscr{K}(\sigma) \pmod{QR_{12}} = \frac{A}{C} X^{n+1} \eta(\sigma) \pmod{\frac{A}{C} X^{n+2} R} = \eta(\sigma) \pmod{QR_{12}}.$$

Therefore, the non-zero class $\kappa = [\mathscr{K} \pmod{QR_{12}}] \in \mathrm{H}^1(K, \chi)$ enjoys the required local description. This finishes the proof. $\square$

Let $\Psi^{\mathrm{univ}} : G_{K,S} \to \Lambda_K^\times$ be the universal character in (2.2). By definition, Ψ is unramified outside $\mathfrak{p}$. For each $\sigma \in G_{K,S}$, we can write $\Psi^{\mathrm{univ}}(\sigma) \equiv 1 + \eta_{\mathfrak{p}}(\sigma) X \pmod{X^2}$ for some $\eta_{\mathfrak{p}} \in \mathrm{Hom}(G_{K,S}^{ab}, \overline{\mathbf{Q}}_p)$. By definition,

$$\boldsymbol{\varepsilon}_{\mathfrak{p}}(\Psi^{\mathrm{univ}}(\mathrm{rec}_{K_p}(u,1))) = \langle u \rangle^{-1} \text{ for } u \in \mathbf{Z}_p^\times. \tag{4.2}$$

Let $\mathbf{v} := \boldsymbol{\varepsilon}_{\mathfrak{p}}(\gamma_0)$ and put

$$\eta_{\overline{\mathfrak{p}}}(\sigma) = \eta_{\mathfrak{p}}(c\sigma c); \quad \eta_v^* = \log_p \mathbf{v} \cdot \eta_v,\ v = \mathfrak{p} \text{ or } \overline{\mathfrak{p}}.$$

Lemma 4.3. *We have*

$$\mathrm{loc}_{\overline{\mathfrak{p}}}(\eta_{\mathfrak{p}}^*) = \mathscr{L}(\mathbf{1}) \cdot \mathrm{ord}_p \quad \text{and} \quad \mathrm{loc}_{\overline{\mathfrak{p}}}(\eta_{\overline{\mathfrak{p}}}^*) = -\log_p - \mathscr{L}(\mathbf{1}) \cdot \mathrm{ord}_p.$$

Proof. Write $\Psi_s(\sigma) := \boldsymbol{\varepsilon}_{\mathfrak{p}}^s(\Psi^{\mathrm{univ}}(\sigma)) = \boldsymbol{\varepsilon}_{\mathfrak{p}}(\sigma)^{-s}$, and then by definition we have

$$\frac{d}{ds} \Psi_s(\sigma)|_{s=0} = \eta_{\mathfrak{p}}(\sigma) \cdot \log_p \mathbf{v} = \eta_{\mathfrak{p}}^*(\sigma) \quad (\boldsymbol{\varepsilon}_{\mathfrak{p}}^s(X) = \mathbf{v}^s - 1). \tag{4.3}$$

Recall that $\mathscr{L}(\mathbf{1}) = \frac{\log_p \overline{\varpi}}{h}$, where h is the class number of K and $\varpi \in K^\times$ with $\mathfrak{p}^h = \varpi \mathcal{O}_K$. Evaluating both sides of (4.3) for $\sigma = \mathrm{rec}_{K_p}(1, \varpi)$, we obtain

$$h \cdot \eta_{\mathfrak{p}}^*(\mathrm{Fr}_{\overline{\mathfrak{p}}}) = \frac{d}{ds} \Psi_s(\mathrm{rec}_{K_p}(1,\varpi))|_{s=0} = \frac{d}{ds} \Psi_s(\mathrm{rec}_{K_p}(\overline{\varpi}^{-1}, 1))|_{s=0} = \log_p \overline{\varpi} = h \cdot \mathscr{L}(\mathbf{1}).$$

Since $\eta_{\mathfrak{p}}^*$ is unramified at $\overline{\mathfrak{p}}$,

$$\mathrm{loc}_{\overline{\mathfrak{p}}}(\eta_{\mathfrak{p}}^*) = \eta_{\mathfrak{p}}^*(\mathrm{Fr}_{\overline{\mathfrak{p}}}) \cdot \mathrm{ord}_p = \mathscr{L}(\mathbf{1}) \cdot \mathrm{ord}_p.$$

From (4.2) and (4.3), we find that

$$\eta_{\overline{\mathfrak{p}}}^*(\mathrm{rec}_{K_p}(1,a)) = \eta_{\mathfrak{p}}^*(\mathrm{rec}_{K_p}(a,1)) = -\log_p a \text{ for } a \in \mathbf{Z}_p^\times.$$

On other hand, $\Psi_s(\mathrm{rec}_{K_p}(\varpi, 1)) = \Psi_s(\mathrm{rec}_{K_p}(1, \overline{\varpi}^{-1})) = 1$, so $\eta_{\overline{\mathfrak{p}}}^*(\mathrm{rec}_{K_p}(1,\varpi)) = \eta_{\mathfrak{p}}^*(\mathrm{rec}_{K_p}(\varpi, 1)) = 0$. These equations imply that

$$\mathrm{loc}_{\overline{\mathfrak{p}}}(\eta_{\overline{\mathfrak{p}}}^*)(a) = \eta_{\overline{\mathfrak{p}}}^*(\mathrm{rec}_{K_p}(1,a)) = -\log_p(a) - \mathscr{L}(\mathbf{1}) \cdot \mathrm{ord}_p(a) \text{ for } a \in \mathbf{Q}_p^\times. \quad \square$$

Theorem 4.4. *We have the following formula for $\mathscr{L}$-invariant:*

$$\mathscr{L}(\chi) = 2\mathscr{L}(\mathbf{1}) - \frac{\mathcal{L}_p'(s, -s, \chi)|_{s=0}}{\mathcal{L}_{\mathfrak{p}}^*(0, \chi)}.$$

Proof. Let $B(s) = \boldsymbol{\varepsilon}_{\mathfrak{p}}^s(\mathcal{C}(\phi^c, \phi)^{-1})$ be as in Corollary 3.8. In view of Lemma 4.1, Lemma 4.3 and the formula Corollary 3.8, it suffices to construct a nonzero element $\kappa \in \mathrm{H}^1(K, \chi)$ such that $\mathrm{loc}_{\overline{\mathfrak{p}}}(\kappa)$ is a nonzero multiple of

$$\mathrm{loc}_{\overline{\mathfrak{p}}}(\eta_{\mathfrak{p}}^*) - \mathrm{loc}_{\overline{\mathfrak{p}}}(\eta_{\overline{\mathfrak{p}}}^*) - \frac{d}{ds} B(s)\Big|_{s=0} \cdot \mathrm{ord}_p = \log_p + \left(2\mathscr{L}(\mathbf{1}) - \frac{d}{ds} B(s)\Big|_{s=0}\right) \cdot \mathrm{ord}_p. \tag{4.4}$$

We shall use Theorem 4.2 and adapt the calculations in [Ven15, §3] to construct such a class. Recall that the Fourier expansion

$$\boldsymbol{\theta}_\phi = \sum_{n=1}^{\infty} \mathbf{a}(n, \boldsymbol{\theta}_\phi) q^n$$

of the Λ-adic CM form $\boldsymbol{\theta}_\phi$ is given by

$$\mathbf{a}(\ell, \boldsymbol{\theta}_\phi) = \begin{cases} \phi\Psi^{\mathrm{univ}}(\mathrm{Fr}_{\mathfrak{l}}) + \phi\Psi^{\mathrm{univ}}(\mathrm{Fr}_{\bar{\mathfrak{l}}}) & \text{if } \ell\mathcal{O}_K = \mathfrak{l}\bar{\mathfrak{l}} \text{ is split,} \\ 0 & \text{if } \ell \text{ is inert,} \\ \phi\Psi^{\mathrm{univ}}(\mathrm{Fr}_{\mathfrak{l}}) & \text{if } \ell \mid d_K C_{\mathrm{s}} \text{ and } \mathfrak{l} \mid (d_K \mathfrak{c}_{\mathrm{s}}, \ell). \end{cases} \tag{4.5}$$

The CM form $\boldsymbol{\theta}_\phi$ is a Λ-adic newform of tame level $N = d_K C_{\mathrm{s}} C_{\mathrm{i}}^2$, so we have

$$T_\ell \boldsymbol{\theta}_\phi = \mathbf{a}(\ell, \boldsymbol{\theta}_\phi)\boldsymbol{\theta}_\phi \text{ if } \ell \nmid N, \quad U_\ell \boldsymbol{\theta}_\phi = \mathbf{a}(\ell, \boldsymbol{\theta}_\phi)\boldsymbol{\theta}_\phi \text{ if } \ell \mid N. \tag{4.6}$$

Consider the Λ-adic cusp form $\mathscr{H} \in \mathbf{S}^\perp$ constructed in (3.8):

$$\mathscr{H} = -\frac{1}{A}\boldsymbol{\theta}_\phi - \frac{1}{B}\boldsymbol{\theta}_{\phi^c} + e_{\mathrm{ord}}(\theta_\phi^\circ \mathcal{G}_C), \text{ where } A = \mathcal{C}(\phi, \phi^c)^{-1} \text{ and } B = \mathcal{C}(\phi^c, \phi)^{-1}.$$

Put

$$b_1 := \frac{B}{X}\Big|_{X=0} = \frac{1}{\log_p \mathbf{v}} \cdot \frac{d}{ds} B(s)\Big|_{s=0}.$$

There are three cases.

Case (i): $\mathrm{ord}_P(A + B) = \mathrm{ord}_P(A)$. Define

$$\mathscr{H}_1 := (-B) \cdot \mathscr{H} \pmod{X^2} = \frac{B}{A} \cdot \boldsymbol{\theta}_\phi + \boldsymbol{\theta}_{\phi^c} - B \cdot e_{\mathrm{ord}}(\mathcal{G}_C \theta_\phi^\circ) \pmod{X^2}.$$

Let $F = \mathcal{W}[\frac{1}{p}]$. Put

$$u_1 = \frac{A}{A+B}\Big|_{X=0} \in F^\times.$$

Then $\mathbf{a}(1, \mathscr{H}_1) \equiv u_1^{-1} \pmod{X}$. Define the additive homomorphism $\psi_1 : G_{K,S} \to F$ by

$$\psi_1 := (1 - u_1)\eta_{\mathfrak{p}} + u_1 \eta_{\overline{\mathfrak{p}}}$$

and the character $\Psi_1 \colon G_{K,S} \to \Lambda_P/(X^2)$ by

$$\Psi_1 = 1 + \psi_1 X \pmod{X^2}.$$

By (4.6), $\mathcal{G}_C \equiv 1 \pmod{X}$ (3.15) and the equations

$$X\boldsymbol{\theta}_\phi \equiv X\boldsymbol{\theta}_{\phi^c} \equiv X\theta_\phi^{(\mathfrak{p})} \pmod{X^2}, \quad X\mathscr{H}_1 = u_1^{-1} X \theta_\phi^{(\mathfrak{p})} \pmod{X^2}; \tag{4.7}$$

$$U_p \theta_\phi^\circ = \phi(\mathfrak{p})\theta_\phi^\circ + \phi(\overline{\mathfrak{p}})\theta_\phi^{(\mathfrak{p})} \quad (\phi(\mathfrak{p}) = \phi(\overline{\mathfrak{p}})), \tag{4.8}$$

we verify that $\mathscr{H}_1$ is an eigenform modulo X^2 with

$$\begin{aligned} T_\ell \mathscr{H}_1 =& (\phi\Psi_1(\mathrm{Fr}_{\mathfrak{l}}) + \phi\Psi_1(\mathrm{Fr}_{\bar{\mathfrak{l}}}))\mathscr{H}_1 \text{ if } \ell \nmid pN \text{ and } \ell\mathcal{O}_K = \mathfrak{l}\bar{\mathfrak{l}} \text{ is split,} \\ U_p \mathscr{H}_1 =& \phi(\overline{\mathfrak{p}})(1 + X(\eta_{\mathfrak{p}}(\mathrm{Fr}_{\overline{\mathfrak{p}}}) - u_1 b_1))\mathscr{H}_1. \end{aligned}$$

Since $\mathscr{H} \in \mathbf{S}^\perp$, this induces a homomorphism $\lambda_{\mathscr{H}} : \mathbf{T}^\perp \to \Lambda_P/(X^2)$ defined by

$$\lambda_{\mathscr{H}}(t) := \mathbf{a}(1, t \cdot \mathscr{H}_1)/\mathbf{a}(1, \mathscr{H}_1) = \mathbf{a}(1, t \cdot \mathscr{H})/\mathbf{a}(1, \mathscr{H}) \pmod{X^2}$$

with

$$\lambda_{\mathscr{H}}(T_\ell) = \phi\Psi_1(\mathrm{Fr}_{\mathfrak{l}}) + \phi\Psi_1(\mathrm{Fr}_{\bar{\mathfrak{l}}}) \text{ if } \ell\mathcal{O}_K = \mathfrak{l}\bar{\mathfrak{l}} \text{ is split,} \ \ \phi(\overline{\mathfrak{p}})^{-1}\lambda_{\mathscr{H}}(U_p) = 1 + X(\eta_{\mathfrak{p}}(\mathrm{Fr}_{\overline{\mathfrak{p}}}) - u_1 b_1).$$

By Theorem 4.2 with $\widetilde{\Psi} = \Psi_1$ and $n = 0$, we find that there exists a nonzero class $\kappa \in \mathrm{H}^1(K, \chi)$ with

$$\mathrm{loc}_{\overline{\mathfrak{p}}}(\kappa) = \mathrm{loc}_{\overline{\mathfrak{p}}}(\eta_{\mathfrak{p}}) - u_1 b_1 \cdot \mathrm{ord}_p - \mathrm{loc}_{\overline{\mathfrak{p}}}(\psi_1) = u_1(\mathrm{loc}_{\overline{\mathfrak{p}}}(\eta_{\mathfrak{p}}) - \mathrm{loc}_{\overline{\mathfrak{p}}}(\eta_{\overline{\mathfrak{p}}}) - b_1 \cdot \mathrm{ord}_p).$$

Case (ii): $\mathrm{ord}_P(A+B) > \mathrm{ord}_P(B) = \mathrm{ord}_P(A)$. In this case, $\mathscr{H}_1$ is not an eigenform of the Hecke algebra $\mathbf{T}$ but a generalized eigenform. Put

$$u_2 = \frac{B}{A}\Big|_{X=0} \in F^{\times}.$$

Define the additive homomorphism $\psi_2 : G_{K,S} \to F$ by

$$\psi_2 := u_2 \eta_{\mathfrak{p}} + \eta_{\overline{\mathfrak{p}}}$$

and define the character $\Psi_2 : G_{K,S} \to \Lambda_K/(X^2)$ by $\Psi_2 = 1 + X\psi_2$. Using the relations (4.6), (4.7) and (4.8), we find that the Hecke algebra $\mathbf{T}$ stabilizes the two-dimensional subspace spanned by $X\theta_{\phi}^{(\mathfrak{p})}$ and $\mathscr{H}_1$. In addition, we have

$$\begin{aligned}
T_\ell \mathscr{H}_1 &= (\phi(\mathfrak{l}) + \phi(\bar{\mathfrak{l}}))\mathscr{H}_1 + (\phi(\mathfrak{l})\psi_2(\mathrm{Fr}_{\mathfrak{l}}) + \phi(\bar{\mathfrak{l}})\psi_2(\mathrm{Fr}_{\bar{\mathfrak{l}}}))X\theta_{\phi}^{(\mathfrak{p})},\\
T_\ell X\theta_{\phi}^{(\mathfrak{p})} &= (\phi(\mathfrak{l}) + \phi(\bar{\mathfrak{l}}))X\theta_{\phi}^{(\mathfrak{p})} \text{ if } \ell \nmid pN \text{ and } \ell\mathcal{O}_K = \mathfrak{l}\bar{\mathfrak{l}} \text{ is split},\\
U_p \mathscr{H}_1 &= \phi(\overline{\mathfrak{p}})\mathscr{H}_1 + \phi(\overline{\mathfrak{p}})((1+u_2)\eta_{\mathfrak{p}}(\mathrm{Fr}_{\overline{\mathfrak{p}}}) - b_1) \cdot X\theta_{\phi}^{(\mathfrak{p})}\\
U_p X\theta_{\phi}^{(\mathfrak{p})} &= \phi(\overline{\mathfrak{p}})X\theta_{\phi}^{(\mathfrak{p})}.
\end{aligned}$$

This yields a homomorphism $\lambda_{\mathscr{H}} : \mathbf{T} \to \mathrm{U} \subset \mathrm{M}_2(\overline{\mathbf{Q}}_p)$, where $\mathrm{U} = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| \, a, b \in \mathcal{W} \right\}$. It is clear that $\lambda_{\mathscr{H}}$ factors through $\mathbf{T}^{\perp}$, and with the identification $\Lambda_K/(X^2) \xrightarrow{\sim} \mathrm{U}$, $X \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we obtain the homomorphism $\lambda_{\mathscr{H}} : \mathbf{T}^{\perp} \to \Lambda_P/(X^2)$ with

$$\begin{aligned}
\lambda_{\mathscr{H}}(T_\ell) &= \phi\Psi_2(\mathrm{Fr}_{\mathfrak{l}}) + \phi\Psi_2(\mathrm{Fr}_{\bar{\mathfrak{l}}}) \text{ if } \ell \nmid pN \text{ and } \ell\mathcal{O}_K = \mathfrak{l}\bar{\mathfrak{l}} \text{ is split},\\
\phi(\overline{\mathfrak{p}})^{-1}\lambda_{\mathscr{H}}(U_p) &= 1 + X\left((1+u_2)\eta_{\mathfrak{p}}(\mathrm{Fr}_{\overline{\mathfrak{p}}}) - b_1\right).
\end{aligned}$$

It follows from Theorem 4.2 that

$$\mathrm{loc}_{\overline{\mathfrak{p}}}(\kappa) = (1+u_2) \cdot \mathrm{loc}_{\overline{\mathfrak{p}}}(\eta_{\mathfrak{p}}) - b_1 \cdot \mathrm{ord}_p - \mathrm{loc}_{\overline{\mathfrak{p}}}(\psi_2) = \mathrm{loc}_{\overline{\mathfrak{p}}}(\eta_{\mathfrak{p}}) - \mathrm{loc}_{\overline{\mathfrak{p}}}(\eta_{\overline{\mathfrak{p}}}) - b_1 \cdot \mathrm{ord}_p.$$

Case (iii): $\mathrm{ord}_P(A) > \mathrm{ord}_P(B) > 0$. Let $n = \mathrm{ord}_P(A/B)$ and

$$u_3 = \frac{A}{BX^n}\Big|_{X=0} \in F^{\times}.$$

Let

$$\mathscr{H}_2 := (-A)\mathscr{H} \pmod{X^{n+2}} = \boldsymbol{\theta}_{\phi} + \frac{A}{B} \cdot \boldsymbol{\theta}_{\phi^c} - A \cdot e_{\mathrm{ord}}(\theta_{\phi}^{\circ}\mathcal{G}_C) \pmod{X^{n+2}}.$$

Then $\mathbf{a}(1, \mathscr{H}_2) \equiv 1 \pmod{X}$. Define the additive homomorphism $\psi_3 : G_K \to F$ by

$$\psi_3 = u_3(\eta_{\overline{\mathfrak{p}}} - \chi^{-1}\eta_{\mathfrak{p}}),$$

and define the character $\Psi_3 : G_{K,S} \to \Lambda_P/(X^{n+2})$ by

$$\Psi_3 = \Psi^{\mathrm{univ}} + \psi_3 X^{n+1} \pmod{X^{n+2}}.$$

Using the equations (4.6), (4.8),

$$\frac{A}{B}\mathscr{H}_2 X \equiv \frac{A}{B}\boldsymbol{\theta}_\phi X \equiv \frac{A}{B}\theta_\phi^{(\mathfrak{p})} X \equiv \frac{A}{B}\boldsymbol{\theta}_{\phi^c} X \pmod{X^{n+2}};$$
$$\left(1-\frac{A}{B}\right)\mathscr{H}_2 X = \boldsymbol{\theta}_\phi X \pmod{X^{n+2}},$$

we can verify that $\mathscr{H}_2$ is an eigenform modulo X^{n+2} and

$$T_\ell \mathscr{H}_2 = (\phi\Psi_3(\mathrm{Fr}_\mathfrak{l}) + \phi\Psi_3(\mathrm{Fr}_{\bar{\mathfrak{l}}}))\mathscr{H}_2 \text{ if } \ell \nmid pN \text{ and } \ell\mathcal{O}_K = \mathfrak{l}\bar{\mathfrak{l}} \text{ is split in } K,$$
$$U_p \mathscr{H}_2 = \phi(\overline{\mathfrak{p}})\left(\Psi^{\mathrm{univ}}(\mathrm{Fr}_{\overline{\mathfrak{p}}}) - b_1 u_3\right) X^{n+1}\mathscr{H}_2 \pmod{X^{n+2}}.$$

Likewise we obtain a homomorphism $\lambda_\mathscr{H} : \mathbf{T}^\perp \to \Lambda_P/(X^{n+2})$ defined by $\lambda_\mathscr{H}(t) = \mathbf{a}(1, t\cdot\mathscr{H}_2)/\mathbf{a}(1,\mathscr{H}_2)$ with

$$\lambda_\mathscr{H}(T_\ell) = \phi\Psi_3(\mathrm{Fr}_\mathfrak{l}) + \phi\Psi_3(\mathrm{Fr}_{\bar{\mathfrak{l}}}) \text{ if } \ell\mathcal{O}_K = \mathfrak{l}\bar{\mathfrak{l}} \text{ is split},$$
$$\phi(\overline{\mathfrak{p}})^{-1}\lambda_\mathscr{H}(U_p) = (\Psi^{\mathrm{univ}}(\mathrm{Fr}_{\overline{\mathfrak{p}}}) - b_1 u_3)X^{n+1}.$$

It follows from Theorem 4.2 that

$$\mathrm{loc}_{\overline{\mathfrak{p}}}(\kappa) = -b_1 u_3 \cdot \mathrm{ord}_p - \mathrm{loc}_{\overline{\mathfrak{p}}}(\psi_3) = u_3(\mathrm{loc}_{\overline{\mathfrak{p}}}(\eta_\mathfrak{p}) - \mathrm{loc}_{\overline{\mathfrak{p}}}(\eta_{\overline{\mathfrak{p}}}) - b_1\cdot \mathrm{ord}_p).$$

In each cases, we see immediately that $\mathrm{loc}_{\overline{\mathfrak{p}}}(\kappa)$ is a multiple of the function in (4.4), and the theorem follows. □

4.3. **Proof of Theorem 1.** We are ready to prove Theorem 1. By Remark 3.5,

$$L_p'(0,\chi) = \left.\frac{L_p(s,\chi)}{s}\right|_{t=0} = \mathcal{L}_p'(s,s,\chi)|_{s=0}.$$

By Theorem 4.4 and Corollary 3.8, we find that the cyclotomic derivative $L_p'(0,\chi)$ equals

$$\begin{aligned}\mathcal{L}_p'(s,s,\chi)|_{s=0} &= 2\mathcal{L}_p'(s,0,\chi)|_{s=0} - \mathcal{L}_p'(s,-s,\chi)|_{s=0}\\ &= 2\mathscr{L}(\mathbf{1})\cdot\mathcal{L}_\mathfrak{p}^*(0,\chi) - \mathcal{L}_p'(s,-s,\chi)|_{s=0}\\ &= \mathcal{L}_\mathfrak{p}^*(0,\chi)\cdot\mathscr{L}(\chi).\end{aligned}$$

Now Theorem 1 follows from (3.5).

5. Comparison of $\mathscr{L}$-invariants

5.1. **Benois' $\mathscr{L}$-invariant.** Here we briefly recall the definition of $\mathscr{L}$-invariant by Benois [Ben11, Ben14, BH20]. Let p be an odd prime. Let $\varepsilon = (\zeta_{p^n})_{n\geq 0}$ be primitive p^n-th roots of unity such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for any $n \geq 0$. We put $K_n = \mathbb{Q}_p(\zeta_{p^n})$ and $K_\infty = \bigcup_{n\geq 0} K_n$. Denote $\Gamma = \mathrm{Gal}(K_\infty/\mathbb{Q}_p)$ and decompose $\Gamma = \Delta\times\Gamma_1$, where $\Gamma_1 = \mathrm{Gal}(K_\infty/K_1)$. Let $\chi_{\mathrm{cyc}} : \Gamma \to \mathbb{Z}_p^\times$ be the cyclotomic character. Let $E/\mathbb{Q}_p$ be a finite extension. For $r \in [0,1)$, we set

$$\mathscr{R}_E^{(r)} = \left\{ f(x) = \sum_{n\in\mathbb{Z}} a_n X^n \;\middle|\; a_n \in E,\ f(X) \text{ converges on } \{X\in\mathbb{C}_p \mid r \leq |X|_p < 1\}\right\}.$$

Then the Robba ring with coefficients in E is defined by $\mathscr{R}_E = \bigcup_{0\leq r<1}\mathscr{R}_E^{(r)}$. The Robba ring $\mathscr{R}_E$ has actions of Γ and a Frobenius operator φ.

For a (φ,Γ)-module $\mathbb{D}$ over the Robba ring $\mathscr{R}_E$, we put $\mathscr{D}_{\mathrm{cris}}(\mathbb{D}) = (\mathbb{D}[1/t])^\Gamma$, where $t = \sum_{n=1}^\infty \frac{X^n}{n}$. For each p-adic representation V of $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, we can associate a (φ,Γ)-module $\mathbb{D}^{\dagger}_{\mathrm{rig}}(V)$. Fix a generator $\gamma_1 \in \Gamma_1$. For any (φ,Γ)-module $\mathbb{D}$, let $H^i(\mathbb{D})$ be the cohomology of Fontaine-Herr complex

$$C_{\varphi,\gamma_1} : \mathbb{D}^{\Delta} \xrightarrow{d_0} \mathbb{D}^{\Delta} \oplus \mathbb{D}^{\Delta} \xrightarrow{d_1} \mathbb{D}^{\Delta},$$

where $d_0(x) = ((\varphi-1)x, (\gamma_1-1)x)$ and $d_1(y,z) = (\gamma_1-1)y-(\varphi-1)z$. Let $\mathbb{D}^*(\chi_{\mathrm{cyc}}) = \mathrm{Hom}_{\mathscr{R}_E}(\mathbb{D}, \mathscr{R}_E(\chi_{\mathrm{cyc}}))$ be the Tate dual. For a (φ,Γ)-module $\mathbb{D}$, define

$$H^1_f(\mathbb{D}) = \left\{\alpha \in H^1(\mathbb{D}) \,\middle|\, D_\alpha \text{ is crystalline}\right\},$$

where D_α is the extension class associated to α.

From now on, we consider the global situation. Fix a finite set of primes S containing p and denote by $\mathbb{Q}_S/\mathbb{Q}$ the maximal Galois extension of $\mathbb{Q}$ unramified outside $S \cup \{\infty\}$. We set $G_{\mathbb{Q},S} = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$. Let V be a p-adic representation of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside S with coefficient in a p-adic field E. Let $H^1_f(\mathbb{Q},V)$ be the Bloch-Kato Selmer group defined by

$$H^1_f(\mathbb{Q},V) = \mathrm{Ker}\left[H^1(G_{\mathbb{Q},S},V) \to \bigoplus_{v\in S} \frac{H^1(\mathbb{Q}_v,V)}{H^1_f(\mathbb{Q}_v,V)}\right].$$

We also denote the relaxed Selmer group by

$$H^1_{f,\{p\}}(\mathbb{Q},V) = \mathrm{Ker}\left[H^1(G_{\mathbb{Q},S},V) \to \bigoplus_{v\in S\smallsetminus\{p\}} \frac{H^1(\mathbb{Q}_v,V)}{H^1_f(\mathbb{Q}_v,V)}\right].$$

We assume the following conditions:
— C1) $H^0(G_{\mathbb{Q},S},V) = H^0(G_{\mathbb{Q},S},V^*(1)) = 0$.
— C2) V is crystalline at p and $D_{\mathrm{cris}}(V)^{\varphi=1} = 0$.
— C3) The action of φ is semisimple on $D_{\mathrm{cris}}(V)$ at p^{-1}.
— C4) $H^1_f(\mathbb{Q},V^*(1)) = 0$.
— C5) $\mathrm{loc}_p : H^1_f(\mathbb{Q},V) \to H^1_f(\mathbb{Q}_p,V)$ is injective.

Definition 5.1. A φ-submodule D of $D_{\mathrm{cris}}(V)$ is regular if $D\cap \mathrm{Fil}^0 D_{\mathrm{cris}}(V) = 0$ and $r_{V,D} : H^1_f(\mathbb{Q},V) \to D_{\mathrm{cris}}(V)/(\mathrm{Fil}^0 D_{\mathrm{cris}}(V)+D)$ is an isomorphism, where $r_{V,D}$ is the map induced by $r_V = \log_V \circ \mathrm{loc}_p : H^1_f(\mathbb{Q},V) \to D_{\mathrm{cris}}(V)/\mathrm{Fil}^0 D_{\mathrm{cris}}(V)$ and $\log_V$ is the Bloch-Kato logarithm.

Let $D \subset D_{\mathrm{cris}}(V)$ be a regular submodule. Then we can decompose $D_0 = D$ into $D = D_{-1} \oplus D^{\varphi=p^{-1}}$ with $D_{-1}^{\varphi=p^{-1}} = 0$. Let $F_0 D^{\dagger}_{\mathrm{rig}}(V)$ and $F_{-1}D^{\dagger}_{\mathrm{rig}}(V)$ be the (φ,Γ)-modules associated to D_0 and D_{-1} by Berger's theory. We set $W = \mathrm{gr}_0 D^{\dagger}_{\mathrm{rig}}(V)$. Assume that all the Hodge-Tate weights are non-negative. Then

$$i_W : \mathscr{D}_{\mathrm{cris}}(W) \oplus \mathscr{D}_{\mathrm{cris}}(W) \to H^1(W)$$

defined by $(x,y) \mapsto \mathrm{cl}(-x, y\log\chi_{\mathrm{cyc}})$ is an isomorphism ([Ben11, Proposition 1.5.9]). Let $i_{W,f}$ and $i_{W,c}$ denote the restriction of i_W on the first and second direct summand respectively. Then we have $\mathrm{Im}(i_{W,f}) = H^1_f(W)$ and a decomposition $H^1(W) = H^1_f(W) \oplus H^1_c(W)$, where $H^1_c(W) = \mathrm{Im}(i_{W,c})$.

For the dual module $W^*(\chi_{\mathrm{cyc}})$, let

$$i_{W^*(\chi_{\mathrm{cyc}})} : \mathscr{D}_{\mathrm{cris}}(W^*(\chi_{\mathrm{cyc}})) \oplus \mathscr{D}_{\mathrm{cris}}(W^*(\chi_{\mathrm{cyc}})) \to H^1(W^*(\chi_{\mathrm{cyc}}))$$

be the unique linear map such that $i_{W^*(\chi_{\rm cyc})}(\alpha,\beta)\cup i_W(x,y)=[\beta,x]_W-[\alpha,y]_W$, where $[\,,\,]_W:\mathscr{D}_{\rm cris}(W^*(\chi_{\rm cyc}))\times\mathscr{D}_{\rm cris}(W)\to E$ denotes the canonical pairing induced by $W^*(\chi_{\rm cyc})\times W\to\mathscr{R}_E(\chi_{\rm cyc})$. Similarly, we can define $i_{W^*(\chi_{\rm cyc}),f}$, $i_{W^*(\chi_{\rm cyc}),c}$ and $H^1_c(W^*(\chi_{\rm cyc}))$ using the map $i_{W^*(\chi_{\rm cyc})}$.

Let

$$\kappa_D: H^1_{f,\{p\}}(\mathbb{Q},V)\to\frac{H^1(\mathbb{Q}_p,V)}{H^1_f(F_0D^\dagger_{\rm rig}(V))}$$

be the composition of the map $\mathrm{loc}_p: H^1_{f,\{p\}}(\mathbb{Q},V)\to H^1(\mathbb{Q}_p,V)$ and the canonical projection. Then κ_D is an isomorphism ([Ben14, Lemma 3.1.4]). We denote

$$H^1(V,D)=\kappa_D^{-1}(H^1(F_0D^\dagger_{\rm rig}(V))/H^1_f(F_0D^\dagger_{\rm rig}(V))).$$

Then the composition of the map $H^1(V,D)\to H^1(F_0D^\dagger_{\rm rig}(V))\to H^1(W)$ induces an isomorphism $H^1(V,D)\simeq H^1(W)/H^1_f(W)$. We consider the following diagram:

$$\begin{tikzcd}
\mathscr{D}_{\rm cris}(W) \arrow[r, "i_{W,f}"] & H^1_f(W) \\
H^1(V,D) \arrow[r] \arrow[u, "\rho_{W,f}"] \arrow[d, "\rho_{W,c}"'] & H^1(W) \arrow[u, "p_{W,f}"'] \arrow[d, "p_{W,c}"] \\
\mathscr{D}_{\rm cris}(W) \arrow[r, "i_{W,c}"] & H^1_c(W),
\end{tikzcd}$$

where $p_{W,f}$ and $p_{W,c}$ are the canonical projections, and $\rho_{W,f}$ and $\rho_{W,c}$ are defined as the unique maps making this diagram commute. Note that $\rho_{W,c}$ is an isomorphism.

Now we define the $\mathscr{L}$-invariant associated to V and D by

$$\mathscr{L}(V,D)=\det\left(\rho_{W,f}\circ\rho_{W,c}^{-1}\,\middle|\,\mathscr{D}_{\rm cris}(W)\right).$$

Remark. In [Ben11], the choice of the sign of the $\mathscr{L}$-invariant is slightly different from [Ben14, BH20]. Here we follow the definition given in [Ben14, BH20].

Next we consider the dual construction of the $\mathscr{L}$-invariant. Let D be a regular submodule of $D_{\rm cris}(V)$ and put

$$D^\perp=D_0^\perp=\mathrm{Hom}_E(D_{\rm cris}(V)/D,D_{\rm cris}(E(1)))$$

and

$$D_1^\perp=\mathrm{Hom}_E(D_{\rm cris}(V)/D_{-1},D_{\rm cris}(E(1))).$$

We denote by $F_0D^\dagger_{\rm rig}(V^*(1))$ (resp. $F_1D^\dagger_{\rm rig}(V^*(1))$) the (φ,Γ)-submodule of $D^\dagger_{\rm rig}(V^*(1))$ associated to $D_0^\perp$ (resp. $D_1^\perp$). Then we have a short exact sequence

$$0\to F_1D^\dagger_{\rm rig}(V^*(1))\to F_0D^\dagger_{\rm rig}(V^*(1))\to W^*(\chi_{\rm cyc})\to 0.$$

Let

$$\kappa_{D^\perp}: H^1_f(\mathbb{Q},V^*(1))\to\frac{H^1(\mathbb{Q}_p,V^*(1))}{H^1_f(\mathbb{Q}_p,V^*(1))+H^1(F_0D^\dagger_{\rm rig}(V^*(1)))}$$

be the map obtained by the composition of loc_p with the canonical projection. We set

$$H^1(V^*(1),D^\perp)=\kappa_{D^\perp}^{-1}\left(H^1(F_1D^\dagger_{\rm rig}(V^*(1)))/(H^1_f(\mathbb{Q}_p,V^*(1))+H^1(F_0D^\dagger_{\rm rig}(V^*(1))))\right).$$

Then the composition of the maps

$$H^1(V^*(1), D^\perp) \to H^1(F_1 D^\dagger_{\mathrm{rig}}(V^*(1))) \to H^1(W^*(\chi_{\mathrm{cyc}}))$$

induces an isomorphism $H^1(V^*(1), D^\perp) \simeq H^1(W^*(\chi_{\mathrm{cyc}}))/H^1_f(W^*(\chi_{\mathrm{cyc}}))$. We consider the following diagram:

$$\begin{tikzcd}
\mathscr{D}_{\mathrm{cris}}(W^*(\chi_{\mathrm{cyc}})) \arrow[r, "i_{W^*(\chi_{\mathrm{cyc}}),f}"] & H^1_f(W^*(\chi_{\mathrm{cyc}})) \\
H^1(V^*(1), D^\perp) \arrow[r] \arrow[u, "\rho_{W^*(\chi_{\mathrm{cyc}}),f}"] \arrow[d, "\rho_{W^*(\chi_{\mathrm{cyc}}),c}"'] & H^1(W^*(\chi_{\mathrm{cyc}})) \arrow[u, "p_{W^*(\chi_{\mathrm{cyc}}),f}"'] \arrow[d, "p_{W^*(\chi_{\mathrm{cyc}}),c}"] \\
\mathscr{D}_{\mathrm{cris}}(W^*(\chi_{\mathrm{cyc}})) \arrow[r, "i_{W^*(\chi_{\mathrm{cyc}}),c}"] & H^1_c(W^*(\chi_{\mathrm{cyc}})),
\end{tikzcd}$$

where $p_{W^*(\chi_{\mathrm{cyc}}),f}$ and $p_{W^*(\chi_{\mathrm{cyc}}),c}$ are the canonical projections, and $\rho_{W^*(\chi_{\mathrm{cyc}}),f}$ and $\rho_{W^*(\chi_{\mathrm{cyc}}),c}$ are defined as the unique maps making this diagram commute. Note that $\rho_{W^*(\chi_{\mathrm{cyc}}),c}$ is an isomorphism.

We define the $\mathscr{L}$-invariant associated to $V^*(1)$ and $D^\perp$ by

$$\mathscr{L}(V^*(1), D^\perp) = (-1)^e \det\left(\rho_{W^*(\chi_{\mathrm{cyc}}),f} \circ \rho^{-1}_{W^*(\chi_{\mathrm{cyc}}),c} \,\Big|\, \mathscr{D}_{\mathrm{cris}}(W^*(\chi_{\mathrm{cyc}}))\right),$$

where $e = \dim_E \mathscr{D}_{\mathrm{cris}}(W^*(\chi_{\mathrm{cyc}}))$.

Proposition 5.2. $\mathscr{L}(V^*(1), D^\perp) = (-1)^e \mathscr{L}(V, D)$.

Proof. See [Ben11, Proposition 2.2.7] and [BH20, Proposition 2.3.8]. □

Using this $\mathscr{L}$-invariant, Benois formulated the exceptional zero conjecture for general crystalline case including non-critical range.

5.2. **Comparison of $\mathscr{L}$-invariants.** Let K be an imaginary quadratic field and p a prime such that $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$. Let $\chi : \mathrm{Gal}(H/K) \to \overline{\mathbb{Q}}_p^\times$ be a non-trivial ring class character. Let E be a p-adic field containing all of the values of χ. Assume that χ is unramified at places above p and $\chi(\overline{\mathfrak{p}}) = 1$. Now we consider the case $V = (\mathrm{Ind}_K^{\mathbb{Q}} \chi)^*(\boldsymbol{\varepsilon}_{\mathrm{cyc}})$. In this case, we have $V^*(1) = \mathrm{Ind}_K^{\mathbb{Q}} \chi$ and it is known that $H^1_f(\mathbb{Q}, V) = H^1_f(K, \chi^{-1}(1)) = (\mathcal{O}_H^\times \otimes E)[\chi]$, $H^1_{f,\{p\}}(\mathbb{Q}, V) = H^1_{\{\mathfrak{p},\overline{\mathfrak{p}}\}}(K, \chi^{-1}(1)) = (\mathcal{O}_H[1/p]^\times \otimes E)[\chi]$ and $H^1_f(\mathbb{Q}, V^*(1)) = H^1_f(K, \chi) = 0$. For $V = (\mathrm{Ind}_K^{\mathbb{Q}} \chi)^*(1)$, it is easy to see that V satisfies the conditions C1) – C5).

Denote

$$V^+ = \{v \in V|_{G_K} \mid \sigma(v) = \chi^{-1}(\sigma)\boldsymbol{\varepsilon}_{\mathrm{cyc}}(\sigma)v \text{ for all } \sigma \in G_K\}$$

and

$$V^- = \{v \in V|_{G_K} \mid \sigma(v) = \chi^{-1}(c\sigma c)\boldsymbol{\varepsilon}_{\mathrm{cyc}}(\sigma)v \text{ for all } \sigma \in G_K\}.$$

Since $\chi \neq \chi^c$, we have a canonical decomposition $V|_{G_K} = V^+ \oplus V^-$. Put $V_{\mathfrak{p}} = V^+|_{G_{K_{\mathfrak{p}}}}$ and $V_{\overline{\mathfrak{p}}} = V^-|_{G_{K_{\overline{\mathfrak{p}}}}}$. Then the natural map $\iota : V|_{\mathbb{Q}_p} \to V_{\mathfrak{p}} \oplus V_{\overline{\mathfrak{p}}}$ becomes an isomorphism. Hence, $H^1(\mathbb{Q}_p, V) = H^1(\mathbb{Q}_p, (\mathrm{Ind}_K^{\mathbb{Q}} \chi)^*(1))$ can be identified with $H^1(K_{\mathfrak{p}}, \chi^{-1}(1)) \oplus H^1(K_{\overline{\mathfrak{p}}}, \chi^{-1}(1))$.

Definition 5.3. We choose a regular submodule D of $D_{\mathrm{cris}}(V)$ as $D = D_{\mathrm{cris}}(\iota^{-1}(V_{\overline{\mathfrak{p}}}))$.

Then $H^1(F_0 D^\dagger_{\mathrm{rig}}(V))$ is identified with $H^1(K_{\overline{\mathfrak{p}}}, \chi^{-1}(1))$ under the isomorphism $H^1(D^\dagger_{\mathrm{rig}}(V)) \simeq H^1(\mathbb{Q}_p, V)$. Here we recall that $F_0 D^\dagger_{\mathrm{rig}}(V)$ is the (φ, Γ)-submodule of $D^\dagger_{\mathrm{rig}}(V)$ associated to D. This property also characterizes the choice of the regular submodule D. Then the modified Euler factor associated to (V, D) is given by

$$\mathcal{E}(V,D) = \det(1-p^{-1}\varphi^{-1}|D)\det(1-\varphi|D_{\mathrm{cris}}(V)/D) = (1-\chi(\overline{\mathfrak{p}}))(1-\chi^{-1}(\mathfrak{p})p^{-1}) = 0$$

and

$$\mathcal{E}^+(V,D) = \det(1-p^{-1}\varphi^{-1}|D_{-1})\det(1-p^{-1}\varphi^{-1}|D^\perp) = (1-\chi(\mathfrak{p})p^{-1}),$$

where $\mathcal{E}^+(V,D)$ is the modified Euler factor which is used in the formula of the exceptional zero conjecture ([Ben14, Conjecture 4]). Note that $\mathcal{E}^+(V,D)$ coincides with the Euler factor appeared in Conjecture 1. Therefore Conjecture 1 is compatible with the exceptional zero conjecture formulated by Benois.

Proposition 5.4. *We have $\mathscr{L}(V,D) = -\mathscr{L}(\chi)$, where $\mathscr{L}(\chi)$ is the $\mathscr{L}$-invariant defined in (1.4).*

Proof. In this case, $F_0 D^\dagger_{\mathrm{rig}}(V) \simeq \mathscr{R}_E(|x|x)$ and $F_{-1} D^\dagger_{\mathrm{rig}}(V) = 0$, where we write x for the character given by the identity map and $|x|$ for $|x| = p^{v_p(x)}$. Hence we have $W = \mathrm{gr}_0\, D^\dagger_{\mathrm{rig}}(V) \simeq \mathscr{R}_E(|x|x)$ and $H^1(W) \simeq H^1(\mathscr{R}_E(|x|x)) \simeq H^1(\mathscr{R}_E(\chi_{\mathrm{cyc}})) \simeq H^1(\mathbb{Q}_p, E(1))$.

Define $\alpha_W = i_{W,f}(1)$ and $\beta_W = i_{W,c}(1)$. Let $\kappa : \mathbb{Q}_p^\times \otimes E \to H^1(\mathbb{Q}_p, E(1))$ be the Kummer map. Then we have $p_{W,f}(\kappa(u)) = \log_p u \cdot \alpha_W$ and $p_{W,c}(\kappa(u)) = \mathrm{ord}_p(u) \cdot \beta_W$ for $u \in \mathbb{Q}_p^\times \otimes E$ (see [Ben11, 1.5.6 and 1.5.10] for details). Since $H^1_{f,\{p\}}(\mathbb{Q}, V) = (\mathcal{O}_H[1/p]^\times \otimes E)[\chi]$ and $H^1(F_0 D^\dagger_{\mathrm{rig}}(V)) = H^1(K_{\overline{\mathfrak{p}}}, \chi^*(1))$, one has

$$\begin{aligned} H^1(V,D) &= \kappa_D^{-1}(H^1(F_0 D^\dagger_{\mathrm{rig}}(V))/H^1_f(F_0 D^\dagger_{\mathrm{rig}}(V))) \\ &= \mathrm{Ker}\left[H^1_{f,\{p\}}(\mathbb{Q}, V) \to H^1(K_{\mathfrak{p}}, \chi^*(1))\right] \\ &= \mathrm{Ker}\left[(\mathcal{O}_H[1/p]^\times \otimes E)[\chi] \to H_{\mathfrak{p}}^\times \otimes E\right]. \end{aligned}$$

In this case, $H^1(V,D)$ is an one-dimensional E-vector space.

Let $\mathrm{ord}_{\overline{\mathfrak{p}}}$ and $\log_{\overline{\mathfrak{p}}}$ be the elements in $H^1(K_{\overline{\mathfrak{p}}}, \mathbb{Q}_p) = \mathrm{Hom}(G_{K_{\overline{\mathfrak{p}}}}, \mathbb{Q}_p)$ corresponding to ord_p and $\log_p$ under the identification $\mathrm{Hom}(G_{K_{\overline{\mathfrak{p}}}}, \mathbb{Q}_p) = \mathrm{Hom}(G_{\mathbb{Q}_p}, \mathbb{Q}_p)$. They can be viewed as maps $\mathrm{ord}_{\overline{\mathfrak{p}}}, \log_{\overline{\mathfrak{p}}} : K_{\overline{\mathfrak{p}}}^\times \to \mathbb{Q}_p$ via the geometrically normalized reciprocity law map $\mathrm{rec}_{\overline{\mathfrak{p}}} : K_{\overline{\mathfrak{p}}}^\times \to G_{K_{\overline{\mathfrak{p}}}}$.

We fix a non-zero element u in the one-dimensional E-vector space $H^1(V,D)$. Then we have

$$\mathscr{L}(V,D) = \frac{\log_{\overline{\mathfrak{p}}}(u)}{\mathrm{ord}_{\overline{\mathfrak{p}}}(u)}$$

by the definition. This shows $\mathscr{L}(V,D) = -\mathscr{L}(\chi)$. □

Next we compute the dual construction. In this case, it is easy to see

$$H^1(F_1 D^\dagger_{\mathrm{rig}}(V^*(1))) = H^1(D^\dagger_{\mathrm{rig}}(V^*(1))) = H^1(\mathbb{Q}_p, V^*(1)) = H^1(K_{\mathfrak{p}}, \chi) \oplus H^1(K_{\overline{\mathfrak{p}}}, \chi),$$

$$H^1(F_0 D^\dagger_{\mathrm{rig}}(V^*(1))) = H^1(K_{\mathfrak{p}}, \chi)$$

and

$$H^1_f(\mathbb{Q}_p, V^*(1)) = H^1_f(K_{\mathfrak{p}}, \chi) \oplus H^1_f(K_{\overline{\mathfrak{p}}}, \chi) = 0.$$

Hence we have

$$H^1(V^*(1),D^\perp)=H^1_{f,\{p\}}(\mathbb{Q},V^*(1))=H^1_{f,\{\mathfrak{p},\overline{\mathfrak{p}}\}}(K,\chi)=H^1(K,\chi),$$

which is an one-dimensional E-vector space. Moreover

$$H^1(W^*(\chi_{\mathrm{cyc}}))\simeq H^1(F_1D^\dagger_{\mathrm{rig}}(V^*(1)))/H^1(F_0D^\dagger_{\mathrm{rig}}(V^*(1)))\simeq H^1(K_{\overline{\mathfrak{p}}},\chi)\simeq H^1(\mathbb{Q}_p,E).$$

Define $\alpha_{W^*(\chi_{\mathrm{cyc}})}=i_{W^*(\chi_{\mathrm{cyc}}),f}(1)$ and $\beta_{W^*(\chi_{\mathrm{cyc}})}=i_{W^*(\chi_{\mathrm{cyc}}),c}(1)$. Under the identification

$$H^1(W^*(\chi_{\mathrm{cyc}}))\simeq H^1(\mathscr{R}_E)\simeq H^1(\mathbb{Q}_p,E),$$

one has $\alpha_{W^*(\chi_{\mathrm{cyc}})}=-\mathrm{ord}_p$ and $\beta_{W^*(\chi_{\mathrm{cyc}})}=\log_p$ (see [Ben11, 1.5.6 and 1.5.10]). Note that our normalization of the reciprocity law map is different from Benois [Ben11, Ben14, BH20]. More precisely, we have $\mathrm{ord}_p(\mathrm{Fr}_p)=1$, where Fr_p is the geometric Frobenius. This gives the difference of the sign with Benois' description.

Fix a non-zero element $\eta\in H^1(V^*(1),D^\perp)=H^1(K,\chi)$. Then we can write

$$\kappa_{D^\perp}(\eta)=x\cdot\mathrm{ord}_{\overline{\mathfrak{p}}}+y\cdot\log_{\overline{\mathfrak{p}}}=(-x)\cdot(-\mathrm{ord}_{\overline{\mathfrak{p}}})+y\cdot\log_{\overline{\mathfrak{p}}}$$

in $H^1(W^*(\chi_{\mathrm{cyc}}))\simeq H^1(K_{\overline{\mathfrak{p}}},\chi)$ and we have $\mathscr{L}(V^*(1),D^\perp)=(-1)^e\left(-\dfrac{x}{y}\right)$, where $e=\dim_E D^{\varphi=p^{-1}}=1$. By Proposition 5.2, we get $\mathscr{L}(V^*(1),D^\perp)=-\mathscr{L}(V,D)=\mathscr{L}(\chi)$ again. Therefore this gives an alternative proof of Lemma 4.1.

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