ON THE NON-VANISHING OF GENERALIZED KATO CLASSES FOR ELLIPTIC CURVES OF RANK TWO

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Abstract. Let E be an elliptic curve over the rationals, and suppose that $L(E,s)$ has sign $+1$ in its functional equation and vanishes at $s=1$. Let $p>3$ be a prime of good ordinary reduction for E. A construction of Darmon–Rotger attaches to E, and an auxiliary weight one cuspidal eigenform g, a Selmer class $\kappa_p \in \mathrm{Sel}(\mathbf{Q}, V_pE)$. Assuming that $L(E, \mathrm{ad}^0(g), 1) \neq 0$, they conjectured that the following are equivalent: (1) $\kappa_p \neq 0$, (2) $\dim_{\mathbf{Q}_p} \mathrm{Sel}(\mathbf{Q}, V_pE) = 2$.

In this paper we prove the Darmon–Rotger conjecture when $\#Ш(E/\mathbf{Q})[p^\infty] < \infty$ (in fact, a weaker condition suffices) and g has CM. The key new ingredient in the proof is a formula for the leading term of a p-adic L-function attached to E in terms of derived p-adic heights, which allows us to realize κ_p as an explicit nonzero multiple of a p-adic regulator constructed from a Mordell–Weil basis (P,Q) of $E(\mathbf{Q}) \otimes \mathbf{Q}$.

Contents

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1. Introduction

Let E be an elliptic curve over $\mathbf{Q}$ (hence modular, [Wil95, TW95, BCDT01]) with associated L-function $L(E,s)$. In the late 1980s, a major advance towards the Birch and Swinnerton-Dyer conjecture was the proof, by Gross–Zagier and Kolyvagin, of the implication

$$\mathrm{ord}_{s=1}L(E,s)=1 \quad \Longrightarrow \quad \mathrm{rank}_{\mathbf{Z}}E(\mathbf{Q})=1 \text{ and } \#\text{Ш}(E/\mathbf{Q})<\infty. \tag{1.1}$$

The proof of (1.1) resorts to choosing an auxiliary imaginary quadratic field $K/\mathbf{Q}$ such that $\mathrm{ord}_{s=1}L(E/K,s)=1$ and for which a Heegner point $y_K\in E(\mathbf{Q})$ can be constructed using the theory of complex multiplication. By the Gross–Zagier formula [GZ86], the non-vanishing of $L'(E/K,1)$ implies that y_K has infinite order, and the proof of (1.1) is reduced to the proof of the implication

$$y_K\notin E(\mathbf{Q})_{\mathrm{tors}} \quad \overset{[\mathrm{Kol88}]}{\Longrightarrow} \quad \mathrm{rank}_{\mathbf{Z}}E(\mathbf{Q})=1 \text{ and } \#\text{Ш}(E/\mathbf{Q})<\infty, \tag{1.2}$$

which was a celebrated theorem by Kolyvagin [Kol88].

A more recent major advance towards the Birch and Swinnerton-Dyer conjecture arises from the works of Kato [Kat04], Skinner–Urban [SU14], Xin Wan [Wan20], and Skinner [Ski20] on the Iwasawa main conjectures for elliptic modular forms, which in particular combine to yield a proof of a p-converse to (1.2):

$$\mathrm{rank}_{\mathbf{Z}}E(\mathbf{Q})=1 \text{ and } \#\text{Ш}(E/\mathbf{Q})[p^\infty]<\infty \quad \overset{[\mathrm{Ski20}]}{\Longrightarrow} \quad y_K\notin E(\mathbf{Q})_{\mathrm{tors}} \tag{1.3}$$

for certain primes p of good ordinary reduction for E. (A slightly different proof of (1.3) was independently found by W. Zhang [Zha14].) When combined with the Gross–Zagier formula, (1.3) yields a p-converse to the Gross–Zagier–Kolyvagin theorem (1.1).

It is natural to ask about the extension of these results to elliptic curves $E/\mathbf{Q}$ of rank $r>1$. As a modest step in this direction, in this paper we prove certain analogues of (1.2) and (1.3) in rank 2, with y_K replaced by a *generalized Kato class*

$$\kappa_p\in \mathrm{Sel}(\mathbf{Q},V_pE)$$

introduced by Darmon–Rotger, [DR17, DR16]. Here $\mathrm{Sel}(\mathbf{Q},V_pE)\subset \mathrm{H}^1(\mathbf{Q},V_pE)$ is the p-adic Selmer group fitting into the exact sequence

$$0\to E(\mathbf{Q})\otimes_{\mathbf{Z}}\mathbf{Q}_p\to \mathrm{Sel}(\mathbf{Q},V_pE)\to T_p\text{Ш}(E/\mathbf{Q})\otimes_{\mathbf{Z}_p}\mathbf{Q}_p\to 0,$$

where $T_p\text{Ш}(E/\mathbf{Q})$ is the p-adic Tate module of the Tate–Shafarevich group $\text{Ш}(E/\mathbf{Q})$.

1.1. **The Darmon–Rotger conjecture.**

We begin by briefly recalling the construction of κ_p by Darmon–Rotger. One starts by associating to the following data:

- a triple of eigenforms $(f,g,h)\in S_2(\Gamma_0(N_f))\times S_1(\Gamma_0(N_g),\chi)\times S_1(\Gamma_0(N_h),\bar{\chi})$ of weights $(2,1,1)$ and level prime-to-p with
$$\gcd(N_f,N_gN_h)=1, \tag{1.4}$$
- a choice of roots $\gamma\in\{\alpha_g,\beta_g\}$ and $\delta\in\{\alpha_h,\beta_h\}$ of the Hecke polynomials of g and h at p, respectively,

a global cohomology class

$$\kappa_{\gamma,\delta}(f,g,h)\in \mathrm{H}^1(\mathbf{Q},V_{fgh}),$$

where $V_{fgh} = V_p(f) \otimes V_p(g) \otimes V_p(h)$ is the tensor product of the p-adic Galois representations associated to f, g and h. Letting $g^\flat$ and $h^\flat$ be the p-stabilizations of g and h with U_p-eigenvalue γ and δ, respectively, this is defined as the *p-adic limit*

$$\kappa_{\gamma,\delta}(f,g,h) := \lim_{\ell\to 1} \kappa(f, \boldsymbol{g}_\ell, \boldsymbol{h}_\ell), \tag{1.5}$$

where $(\boldsymbol{g}_\ell, \boldsymbol{h}_\ell)$ ranges over the classical weight $\ell \geqslant 2$ specializations of Hida families $\boldsymbol{g}$ and $\boldsymbol{h}$ passing through $g^\flat$ and $h^\flat$, respectively, in weight 1, and $\kappa(f, \boldsymbol{h}_\ell, \boldsymbol{h}_\ell)$ is obtained from the p-adic étale Abel–Jacobi image of generalized Gross–Kudla–Schoen diagonal cycles, [GK92, GS95], on a triple product of Kuga–Sato varieties fibered over modular curves.

Remark 1.1. Under assumption (1.4) on the levels, the sign in the functional equation for the triple product L-series $L(s, f \otimes \boldsymbol{g}_\ell \otimes \boldsymbol{h}_\ell)$ is -1 for all $\ell \geqslant 2$; in particular, $L(1, f \otimes \boldsymbol{g}_\ell \otimes \boldsymbol{h}_\ell) = 0$, and by the Gross–Zagier formula for diagonal cycles (proved in [YZZ12] for $\ell = 2$), the classes $\kappa(f, \boldsymbol{g}_\ell, \boldsymbol{h}_\ell)$ should be non-trivial precisely when $L'(1, f \otimes \boldsymbol{g}_\ell \otimes \boldsymbol{h}_\ell) \neq 0$. On the other hand, the global root number of $L(s, f \otimes g \otimes h)$ is $+1$, and it is precisely this *sign-change* phenomenon between weight $\ell \geqslant 2$ and $\ell = 1$ that makes it possible for the p-adic limit construction (1.5) to yield interesting cohomology classes in situations of *even* analytic rank; in fact, as we recall below, classes that are crystalline at p precisely when $\mathrm{ord}_{s=1} L(s, f \otimes g \otimes h) \geqslant 2$.

Assuming $p > 3$ is a prime of good ordinary reduction for f, the explicit reciprocity law of [DR17] yields a formula of the form

$$\exp_p^*(\kappa_{\gamma,\delta}(f,g,h)) = L(1, f \otimes g \otimes h) \cdot (\text{nonzero constant}), \tag{1.6}$$

where $\exp_p^* : \mathrm{H}^1(\mathbf{Q}, V_{fgh}) \to \mathbf{Q}_p$ is the composition of the restriction map $\mathrm{Loc}_p : \mathrm{H}^1(\mathbf{Q}, V_{fgh}) \to \mathrm{H}^1(\mathbf{Q}_p, V_{fgh})$ with the Bloch–Kato dual exponential map (paired against a differential attached to (f, g, h)). In particular, the class $\kappa_{\gamma,\delta}(f,g,h)$ is crystalline at p, and therefore lands in the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V_{fgh}) \subset \mathrm{H}^1(\mathbf{Q}, V_{fgh})$, precisely when $L(s, f \otimes g \otimes h)$ vanishes at $s = 1$.

With the different choices for γ and δ, one thus obtains up to four *a priori* distinct classes $\kappa_{\gamma,\delta}(f,g,h) \in \mathrm{Sel}(\mathbf{Q}, V_{fgh})$ whenever $L(1, f \otimes g \otimes h) = 0$, which Darmon–Rotger conjectured to span a non-trivial subspace of $\mathrm{Sel}(\mathbf{Q}, V_{fgh})$ if and only if $\mathrm{Sel}(\mathbf{Q}, V_{fgh})$ is two-dimensional. In particular, this construction of $\kappa_{\gamma,\delta}(f,g,h)$ yields Selmer classes with a bearing on the arithmetic of elliptic curves $E/\mathbf{Q}$ by taking f to be the newform associated to E, and $h = g^*$ to be the dual of g, so that the triple tensor product V_{fgh} decomposes as

$$V_{fgh} \simeq V_pE \oplus \big(V_pE \otimes \mathrm{ad}^0 V_p(g)\big), \tag{1.7}$$

where $\mathrm{ad}^0 V_p(g)$ is the three-dimensional $G_{\mathbf{Q}}$-representation on the space of trace zero endomorphisms of $V_p(g)$. Correspondingly, $L(s, f \otimes g \otimes h)$ factors as

$$L(s, f \otimes g \otimes h) = L(E, s) \cdot L(E, \mathrm{ad}^0(g), s).$$

In particular, by (1.6), whenever $L(E, 1) = 0$ the above construction yields the four *generalized Kato classes*

$$\kappa_{\alpha_g,\alpha_g^{-1}}(f,g,g^*), \quad \kappa_{\alpha_g,\beta_g^{-1}}(f,g,g^*), \quad \kappa_{\beta_g,\alpha_g^{-1}}(f,g,g^*), \quad \kappa_{\beta_g,\beta_g^{-1}}(f,g,g^*) \tag{1.8}$$

in the Selmer group

$$\mathrm{Sel}(\mathbf{Q}, V_{fgh}) \simeq \mathrm{Sel}(\mathbf{Q}, V_pE) \oplus \mathrm{Sel}(\mathbf{Q}, V_pE \otimes \mathrm{ad}^0 V_p(g)).$$

Assuming that $L(E, \mathrm{ad}^0(g), 1) \neq 0$ (which implies that $\mathrm{Sel}(\mathbf{Q}, V_pE \otimes \mathrm{ad}^0 V_p(g)) = 0$ by the Bloch–Kato conjecture), the non-vanishing criterion conjectured in [DR16, Conj. 3.2] leads to the following prediction (see the "adjoint rank $(2, 0)$ setting" discussed in [DR17, §4.5.3]).

Conjecture 1.2 (Darmon–Rotger)**.** *Suppose that $L(E, s)$ has sign $+1$ and vanishes at $s = 1$, and that $L(E, \mathrm{ad}^0(g), 1) \neq 0$. Then the following are equivalent:*

(i) *The four classes in* (1.8) *span a non-trivial subspace of* $\mathrm{Sel}(\mathbf{Q}, V_pE)$.
(ii) $\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2$.

Remark 1.3. Of course, by the Birch and Swinnerton-Dyer conjecture, condition (ii) should be equivalent to the condition $\mathrm{ord}_{s=1}L(E, s) = 2$, but unfortunately this still seems completely out of reach. More generally, [DR16, Conj. 3.2] posits a similar non-vanishing criterion for the span of the classes $\kappa_{\gamma,\delta}(f, g, h)$ attached to any triple (f, g, h) as above, but Conjecture 1.2 encompasses all the cases of relevance for the study of elliptic curves $E/\mathbf{Q}$ of rank 2.

Note that Conjecture 1.2 does not predict that the four classes in (1.8) generate $\mathrm{Sel}(\mathbf{Q}, V_pE)$. In fact, a strengthtening of the *elliptic Stark conjectures* in [DLR15] predicts that in the setting of Conjecture 1.2 only the classes $\kappa_{\alpha_g,\alpha_g^{-1}}(f, g, g^*)$ and $\kappa_{\beta_g,\beta_g^{-1}}(f, g, g^*)$ are nonzero, and they are the same class up to a nonzero algebraic constant. Our results also provide evidence for this remarkable prediction (see Remark 1.5 below and §5.6 for further details).

1.2. **Statement of the main result.** In this paper we prove Conjecture 1.2 in the case when g has CM, assuming $\#\text{Ш}(E/\mathbf{Q})[p^\infty] < \infty$ (in fact, a weaker condition suffices) for one of the implications.

As before, let $E/\mathbf{Q}$ be an elliptic curve with good ordinary reduction at $p > 3$, and let $f \in S_2(\Gamma_0(N_f))$ be the associated newform. Let K be an imaginary quadratic field of discriminant prime of N_f in which $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ splits, and let ψ be a ray class character of K of conductor prime to pN_f valued in a number field L. The weight one theta series $g = \theta_\psi$ then satisfies

$$L(E, \mathrm{ad}^0(g), s) = L(E^K, s) \cdot L(E/K, \chi, s),$$

where E^K is the twist of E by the quadratic character associated to K, and χ is the ring class character given by ψ/ψ^τ, for ψ^τ the composition of ψ with the action of complex conjugation. Clearly, in this case we may take $\alpha_g = \psi(\overline{\mathfrak{p}})$ and $\beta_g = \psi(\mathfrak{p})$, which we shall simply denote by α and β, respectively, and g^* is the theta series of ψ^{-1}. As in the formulation of the conjectures in [DR16], we assume that $\alpha_g \neq \beta_g$, i.e., $\chi(\overline{\mathfrak{p}}) \neq 1$.

Let $\bar{\rho}_{E,p} : G_{\mathbf{Q}} \to \mathrm{Aut}_{\mathbf{F}_p}(E[p])$ the mod p representation associated to E, and denote by N_f^- the largest factor of N_f divisible only by primes that are inert in K. Finally, let

$$\mathrm{Loc}_p : \mathrm{Sel}(\mathbf{Q}, V_pE) \to \mathrm{H}^1(\mathbf{Q}_p, V_pE)$$

be the restriction map at p.

Theorem A. *Suppose that $L(E, s)$ has sign $+1$ and vanishes at $s = 1$, and that $L(E^K, 1) \cdot L(E/K, \chi, 1) \neq 0$. Suppose also that:*
(a) *$\bar{\rho}_{E,p}$ is irreducible,*
(b) *N_f^- is squarefree,*
(c) *$\bar{\rho}_{E,p}$ is ramified at every prime $q|N_f^-$.*

Then $\kappa_{\alpha,\beta^{-1}}(f, g, g^*) = \kappa_{\beta,\alpha^{-1}}(f, g, g^*) = 0$, and the following hold:

$$\kappa_{\alpha,\alpha^{-1}}(f, g, g^*) \neq 0 \quad\Longrightarrow\quad \dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2, \tag{1.9}$$

and conversely,

$$\left.\begin{array}{l}\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2\\ \mathrm{Sel}(\mathbf{Q}, V_pE) \neq \ker(\mathrm{Loc}_p)\end{array}\right\} \quad\Longrightarrow\quad \kappa_{\alpha,\alpha^{-1}}(f, g, g^*) \neq 0. \tag{1.10}$$

In particular, if $\mathrm{Sel}(\mathbf{Q}, V_p) \neq \ker(\mathrm{Loc}_p)$ then Conjecture 1.2 holds.

If $L(E, s)$ has sign $+1$ and $\bar{\rho}_{E,p}$ is irreducible and ramified at some prime $q \neq p$ (as is automatic if e.g. E is semistable and $p \geqslant 11$ is good ordinary for E, by [Rib90] and [Maz78]), the non-vanishing results of [BFH90] and [Vat03] assure the existence of infinitely many imaginary quadratic fields K and ring class characters χ such that $L(E^K, 1) \cdot L(E/K, \chi, 1) \neq 0$.

Therefore, Theorem A suggests a general construction of non-trivial p-adic Selmer classes for elliptic curves of rank two.

Remark 1.4. The condition $\mathrm{Sel}(\mathbf{Q}, V_pE) \neq \ker(\mathrm{Loc}_p)$ should always hold when $\mathrm{Sel}(\mathbf{Q}, V_pE) \neq 0$. Indeed, if $\mathrm{Sel}(\mathbf{Q}, V_pE)$ equals $\ker(\mathrm{Loc}_p)$, then $E(\mathbf{Q})$ must be finite (since $E(\mathbf{Q})$ injects into $E(\mathbf{Q}_p)$), so if also $\mathrm{Sel}(\mathbf{Q}, V_pE) \neq 0$ we would conclude that $Ш(E/\mathbf{Q})[p^\infty]$ is infinite.

Remark 1.5. It also follows from our results that, for $g = \theta_\psi$ as above, the classes $\kappa_{\alpha,\alpha^{-1}}(f, g, g^*)$ and $\kappa_{\beta,\beta^{-1}}(f, g, g^*)$ are the same up to a nonzero algebraic constant, and they span the *p-adic* line

$$\mathscr{L}_p := \ker(\log_p) \subset \mathrm{Sel}(\mathbf{Q}, V_pE),$$

where $\log_p : \mathrm{Sel}(\mathbf{Q}, V_pE) \to \mathbf{Q}_p$ is the composition of Loc_p with the formal group logarithm of E. When $\#Ш(E/\mathbf{Q})[p^\infty] < \infty$, it is suggestive to view $\mathscr{L}_p$ as the line spanned by the image of $P \wedge Q := P \otimes Q - Q \otimes P \in \bigwedge^2(E(\mathbf{Q}) \otimes \mathbf{Q})$ under the natural map

$$\mathrm{Log}_p : \bigwedge^2(E(\mathbf{Q}) \otimes \mathbf{Q}) \to E(\mathbf{Q}) \otimes \mathbf{Q}_p$$

induced by the p-adic logarithm map $\log_p : E(\mathbf{Q}) \otimes \mathbf{Q} \to E(\mathbf{Q}_p) \otimes \mathbf{Q} \to \mathbf{Q}_p$. This is consistent with the refined predictions by Darmon–Rotger (see [DR16, §4.5.3]), and substantiates viewing implications (1.9) and (1.10) in Theorem A as counterparts of (1.2) and (1.3), respectively, in rank 2.

Remark 1.6. Assuming $\mathrm{rank}_{\mathbf{Z}} E(\mathbf{Q}) = 2$ and the finiteness of $\#Ш(E/\mathbf{Q})[p^\infty]$, a refinement of Conjecture 1.2 predicting the *position* of $\kappa_{\gamma,\delta}(f, g, g^*)$ relative to the natural rational structure on $\mathrm{Sel}(\mathbf{Q}, V_pE) = E(\mathbf{Q}) \otimes \mathbf{Q}_p$ leads to the expectation

$$\kappa_{\alpha,\alpha^{-1}}(f, g, g^*) \overset{?}{\sim}_{\overline{\mathbf{Q}}^\times} \mathrm{Log}_p(P \wedge Q) \overset{?}{\sim}_{\overline{\mathbf{Q}}^\times} \kappa_{\beta,\beta^{-1}}(f, g, g^*) \tag{1.11}$$

where (P, Q) is any basis for $E(\mathbf{Q}) \otimes \mathbf{Q}$ and $\sim_{\overline{\mathbf{Q}}^\times}$ denotes equality up to multiplication by an non-zero algebraic number. Our methods confirm the relation $\kappa_{\alpha,\alpha^{-1}}(f, g, g^*) \sim_{\overline{\mathbf{Q}}^\times} \kappa_{\beta,\beta^{-1}}(f, g, g^*)$ and in Theorem 5.5, we show that

$$\kappa_{\alpha,\alpha^{-1}}(f, g, g^*) \sim_{\mathbf{Q}^\times} C \cdot \mathrm{Log}_p(P \wedge Q), \tag{1.12}$$

where $C \in \mathbf{Q}_p^\times$ is the ratio between the the leading coefficient of the anticyclotomic p-adic L-function of E/K and the derived p-adic height pairing of P and Q. In particular, this implies that the conjectured algebraicity in (1.11) can be linked to a p-adic Birch and Swinnerton-Dyer conjecture refining [BD96, Conjecture 4.3] (see §5.6 for details).

The essential new ingredient in the proof of Theorem A is a formula for the *leading term* at $T = 0$ of an anticyclotomic p-adic L-function $\Theta_{f/K} \in \mathbf{Z}_p[\![T]\!]$ attached to E/K in terms of anticyclotomic *derived p-adic heights* (see Theorem 5.3). This leading term formula also leads to the expression for $\kappa_{\alpha,\alpha^{-1}}(f, g, g^*)$ yielding (1.12) and is used in the Appendix of this paper to exhibit the first examples of non-vanishing generalized Kato classes for elliptic curves E over $\mathbf{Q}$ of rank two, answering a question (or "an interesting challenge"; see [DR16, p. 31]) posed by Darmon–Rotger.

1.3. **Relation to previous work.** Prior to this paper, the only general results known to the present authors on the existence on nonzero Selmer classes for elliptic curves $E/\mathbf{Q}$ of rank $r > 1$ are those to appear in forthcoming work by Skinner–Urban, as reported on in [Urb13]. Their methods, which extend those outlined in their ICM address [SU06] for cuspidal eigenforms of weight $k \geqslant 4$, are completely different form ours.

On the other hand, the celebrated work of Darmon–Rotger [DR17] exhibited, under a *non-vanishing* hypothesis, the existence of two linearly independent classes in the Selmer groups $\mathrm{Sel}(\mathbf{Q}, V_pE \otimes \varrho)$ of elliptic curves $E/\mathbf{Q}$ twisted by degree four Artin representations ϱ. The required non-vanishing is that of a special value $\mathscr{L}_p^{g_\alpha}$ of a certain p-adic L-function playing the role of a second derivative. Both their works and ours exploit the construction of generalized Kato classes introduced in [DR17], but in the setting we have placed ourselves in, the special value $\mathscr{L}_p^{g_\alpha}$ *vanishes*. Our analysis in this paper fundamentally exploits anticyclotomic Iwasawa theory and derived p-adic heights, both of which make no appearance in [DR17].

Finally, as alluded to above, a key ingredient in the proof of our main results is a leading term formula for $\Theta_{f/K}$ in terms of anticyclotomic derived p-adic heights. In the cyclotomic setting, and for the usual p-adic height pairing, a formula of this sort for the first derivative of a p-adic L-function is due to Rubin [Rub94]. An abstract generalization of Rubin's formula for derived p-adic heights was given by Howard [How04] in terms of a cohomologically defined "p-adic L-function". Howard's foundational results on derived p-adic heights will be our starting point in §4, which, as far as we know, contains the first explicit computation of a generalized Rubin formula for genuinely derived p-adic heights.

2. Triple products and theta elements

In this section we describe the triple product p-adic L-function for Hida families [Hsi21], and recall its relation with the square-root anticyclotomic p-adic L-functions of Bertolini–Darmon [BD96].

2.1. **Ordinary Λ-adic forms.** Fix a prime $p > 2$. Let $\mathbb{I}$ be a normal domain finite flat over $\Lambda := \mathcal{O}[\![1 + p\mathbf{Z}_p]\!]$, where $\mathcal{O}$ is the ring of integers of a finite extension $L/\mathbf{Q}_p$. We say that a point $x \in \operatorname{Spec} \mathbb{I}(\overline{\mathbf{Q}}_p)$ is *locally algebraic* if its restriction to $1+p\mathbf{Z}_p$ is given by $x(\gamma) = \gamma^{k_x}\epsilon_x(\gamma)$ for some integer k_x, called the *weight* of x, and some finite order character $\epsilon_x : 1+p\mathbf{Z}_p \to \mu_{p^\infty}$; we say that x is *arithmetic* if it has weight $k_x \geqslant 2$. Let $\mathfrak{X}_{\mathbb{I}}^+$ be the set of arithmetic points.

Fix a positive integer N prime to p, and let $\chi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}^\times$ be a Dirichlet character modulo Np. Let $S^o(N, \chi, \mathbb{I})$ be the space of *ordinary* $\mathbb{I}$-*adic cusp forms* of tame level N and branch character χ, consisting of formal power series

$$\boldsymbol{f}(q) = \sum_{n=1}^{\infty} a_n(\boldsymbol{f})q^n \in \mathbb{I}[\![q]\!]$$

such that for every $x \in \mathfrak{X}_{\mathbb{I}}^+$ the specialization $\boldsymbol{f}_x(q)$ is the q-expansion of a p-ordinary cusp form $\boldsymbol{f}_x \in S_{k_x}(Np^{r_x+1}, \chi\omega^{2-k_x}\epsilon_x)$. Here $r_x \geqslant 0$ is such that $\epsilon_x(1+p)$ has exact order p^{r_x}, and $\omega : (\mathbf{Z}/p\mathbf{Z})^\times \to \mu_{p-1}$ is the Teichmüller character.

We say that $\boldsymbol{f} \in S^o(N, \chi, \mathbb{I})$ is a *primitive Hida family* if for every $x \in \mathfrak{X}_{\mathbb{I}}^+$ we have that $\boldsymbol{f}_x$ is an ordinary p-stabilized newform (in the sense of [Hsi21, Def. 2.4]) of tame level N. Given a primitive Hida family $\boldsymbol{f} \in S^o(N, \chi, \mathbb{I})$, and writing $\chi = \chi'\chi_p$ with χ' (resp. χ_p) a Dirichlet modulo N (resp. p), there is a primitive $\boldsymbol{f}^\iota \in S^o(N, \chi_p\overline{\chi}', \mathbb{I})$ with Fourier coefficients

$$a_\ell(\boldsymbol{f}^\iota) = \begin{cases} \overline{\chi}'(\ell)a_\ell(\boldsymbol{f}) & \text{if } \ell \nmid N, \\ a_\ell(\boldsymbol{f})^{-1}\chi_p\omega^2(\ell)\langle\ell\rangle_{\mathbb{I}}\ell^{-1} & \text{if } \ell \mid N, \end{cases}$$

having the property that for every $x \in \mathfrak{X}_{\mathbb{I}}^+$ the specialization $\boldsymbol{f}_x^\iota$ is the p-stabilized newform attached to the character twist $\boldsymbol{f}_x \otimes \overline{\chi}'$. Let $T^o(N, \chi, \mathbb{I})$ be the $\mathbb{I}$-algebra generated by Hecke operators acting on $S^0(N, \chi, \mathbb{I})$ and let $\lambda_{\boldsymbol{f}} : T^o(N, \chi, \mathbb{I}) \to \mathbb{I}$ be the $\mathbb{I}$-algebra homomoprhism induced by $\boldsymbol{f}$. Let $C(\lambda_{\boldsymbol{f}})$ be the congruence module associated with $\lambda_{\boldsymbol{f}}$ ([Hid88, (5.1)]) and let $\eta_{\boldsymbol{f}} := \operatorname{Ann}_{\mathbb{I}}(C(\lambda_{\boldsymbol{f}}))$ be the congruence ideal of $\boldsymbol{f}$.

By [Hid86] (*cf.* [Wil88, Thm. 2.2.1]), attached to every primitive Hida family $\boldsymbol{f} \in S^o(N, \chi, \mathbb{I})$ there is a continuous $\mathbb{I}$-adic representation $\rho_{\boldsymbol{f}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\operatorname{Frac} \mathbb{I})$ which is unramified outside Np and such that for every prime $\ell \nmid Np$,

$$\operatorname{tr} \rho_{\boldsymbol{f}}(\mathrm{Frob}_\ell) = a_\ell(\boldsymbol{f}), \quad \det \rho_{\boldsymbol{f}}(\mathrm{Frob}_\ell) = \chi\omega^2(\ell)\langle\ell\rangle_{\mathbb{I}}\ell^{-1},$$

where $\langle \ell \rangle_{\mathbb{I}} \in \mathbb{I}^\times$ is the image of $\langle \ell \rangle := \ell\omega^{-1}(\ell) \in 1 + p\mathbf{Z}_p$ under the natural map $1 + p\mathbf{Z}_p \to \mathcal{O}[\![1 + p\mathbf{Z}_p]\!]^\times = \Lambda^\times \to \mathbb{I}^\times$. In particular, letting $\langle \varepsilon_{\rm cyc} \rangle_{\mathbb{I}} : G_{\mathbf{Q}} \to \mathbb{I}^\times$ be defined by $\langle \varepsilon_{\rm cyc} \rangle_{\mathbb{I}}(\sigma) = \langle \varepsilon_{\rm cyc}(\sigma) \rangle_{\mathbb{I}}$, it follows that $\rho_{\boldsymbol{f}}$ has determinant $\chi_{\mathbb{I}}^{-1}\varepsilon_{\rm cyc}^{-1}$, where $\chi_{\mathbb{I}} : G_{\mathbf{Q}} \to \mathbb{I}^\times$ is given by $\chi_{\mathbb{I}} := \sigma_\chi \langle \varepsilon_{\rm cyc} \rangle^{-2} \langle \varepsilon_{\rm cyc} \rangle_{\mathbb{I}}$, with σ_χ the Galois character sending $\mathrm{Frob}_\ell \mapsto \chi(\ell)^{-1}$. Moreover, by [Wil88, Thm. 2.2.2] the restriction of $\rho_{\boldsymbol{f}}$ to $G_{\mathbf{Q}_p}$ is given by

$$\rho_{\boldsymbol{f}}|_{G_{\mathbf{Q}_p}} \sim \begin{pmatrix} \psi_{\boldsymbol{f}} & * \\ 0 & \psi_{\boldsymbol{f}}^{-1}\chi_{\mathbb{I}}^{-1}\varepsilon_{\rm cyc}^{-1} \end{pmatrix} \tag{2.1}$$

where $\psi_{\boldsymbol{f}} : G_{\mathbf{Q}_p} \to \mathbb{I}^\times$ is the unramified character with $\psi_{\boldsymbol{f}}(\mathrm{Frob}_p) = a_p(\boldsymbol{f})$.

2.2. **Triple product p-adic L-function.** Let

$$(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in S^o(N_{\boldsymbol{f}}, \chi_{\boldsymbol{f}}, \mathbb{I}_{\boldsymbol{f}}) \times S^o(N_{\boldsymbol{g}}, \chi_{\boldsymbol{g}}, \mathbb{I}_{\boldsymbol{g}}) \times S^o(N_{\boldsymbol{h}}, \chi_{\boldsymbol{h}}, \mathbb{I}_{\boldsymbol{h}})$$

be a triple of primitive Hida families. Set

$$\mathcal{R} := \mathbb{I}_{\boldsymbol{f}} \hat{\otimes}_{\mathcal{O}} \mathbb{I}_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}} \mathbb{I}_{\boldsymbol{h}},$$

which is a finite extension of the three-variable Iwasawa algebra $\mathcal{R}_0 := \Lambda\hat{\otimes}_{\mathcal{O}}\Lambda\hat{\otimes}_{\mathcal{O}}\Lambda$, and define the weight space $\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$ for the triple $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ in the $\boldsymbol{f}$*-dominated unbalanced range* by

$$\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}} := \Big\{ (x, y, z) \in \mathfrak{X}_{\mathbb{I}_{\boldsymbol{f}}}^+ \times \mathfrak{X}_{\mathbb{I}_{\boldsymbol{g}}}^{\rm cls} \times \mathfrak{X}_{\mathbb{I}_{\boldsymbol{h}}}^{\rm cls} \; : \; k_x \geqslant k_y + k_z \text{ and } k_x \equiv k_y + k_z \pmod 2 \Big\}, \tag{2.2}$$

where $\mathfrak{X}_{\mathbb{I}_{\boldsymbol{g}}}^{\rm cls} \supset \mathfrak{X}_{\mathbb{I}_{\boldsymbol{g}}}^+$ (and similarly $\mathfrak{X}_{\mathbb{I}_{\boldsymbol{h}}}^{\rm cls}$) is the set of locally algebraic points in Spec $\mathbb{I}_{\boldsymbol{g}}(\overline{\mathbf{Q}}_p)$ for which $\boldsymbol{g}_x(q)$ is the q-expansion of a classical modular form.

For $\boldsymbol{\phi} \in \{\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}\}$ and a positive integer N prime to p and divisible by $N_{\boldsymbol{\phi}}$, define the space of Λ-*adic test vectors* $S^o(N, \chi_{\boldsymbol{\phi}}, \mathbb{I}_{\boldsymbol{\phi}})[\boldsymbol{\phi}]$ of level N to be the $\mathbb{I}_{\boldsymbol{\phi}}$-submodule of $S^o(N, \chi_{\boldsymbol{\phi}}, \mathbb{I}_{\boldsymbol{\phi}})$ generated by $\{\boldsymbol{\phi}(q^d)\}$, as d ranges over the positive divisors of $N/N_{\boldsymbol{\phi}}$.

For the next result, set $N := \mathrm{lcm}(N_{\boldsymbol{f}}, N_{\boldsymbol{g}}, N_{\boldsymbol{h}})$, and consider the following hypothesis:

$$\text{for some } (x, y, z) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}, \text{ we have } \varepsilon_q(\boldsymbol{f}_x^\circ, \boldsymbol{g}_y^\circ, \boldsymbol{h}_z^\circ) = +1 \text{ for all } q \mid N. \tag{$\Sigma^- = \emptyset$}$$

Here $\varepsilon_q(\boldsymbol{f}_x^\circ, \boldsymbol{g}_y^\circ, \boldsymbol{h}_z^\circ)$ denotes the local root number of the Kummer self-dual twist of the Galois representations attached to the newforms $\boldsymbol{f}_x^\circ$, $\boldsymbol{g}_y^\circ$, and $\boldsymbol{h}_z^\circ$ corresponding to $\boldsymbol{f}_x$, $\boldsymbol{g}_y$, and $\boldsymbol{h}_z$.

Theorem 2.1. *In addition to the condition* $(\Sigma^- = \emptyset)$*, assume that the triple* $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ *satisfies*

- (ev) $\chi_{\boldsymbol{f}}\chi_{\boldsymbol{g}}\chi_{\boldsymbol{h}} = \omega^{2a}$ *for some* $a \in \mathbf{Z}$,
- (sq) $\gcd(N_{\boldsymbol{f}}, N_{\boldsymbol{g}}, N_{\boldsymbol{h}})$ *is squarefree.*

Then there exist Λ*-adic test vectors* $(\breve{\underline{\boldsymbol{f}}}, \breve{\underline{\boldsymbol{g}}}, \breve{\underline{\boldsymbol{h}}})$ *and an element*

$$\mathscr{L}_p^f(\breve{\underline{\boldsymbol{f}}}, \breve{\underline{\boldsymbol{g}}}, \breve{\underline{\boldsymbol{h}}}) \in \mathcal{R} \otimes_{\mathbb{I}} \mathrm{Frac}\, \mathbb{I}_{\boldsymbol{f}}$$

such that $H \cdot \mathscr{L}_p^f(\breve{\underline{\boldsymbol{f}}}, \breve{\underline{\boldsymbol{g}}}, \breve{\underline{\boldsymbol{h}}}) \in \mathcal{R}$ *for any* $H \in \eta_{\boldsymbol{f}}$ *and that for all* $(x, y, z) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$ *of weight* (k, ℓ, m)*:*

$$\mathscr{L}_p^f(\breve{\underline{\boldsymbol{f}}}, \breve{\underline{\boldsymbol{g}}}, \breve{\underline{\boldsymbol{h}}})(x, y, z)^2 = \frac{\Gamma(k, \ell, m)}{2^{\alpha(k,\ell,m)}} \cdot \frac{\mathcal{E}(\boldsymbol{f}_x, \boldsymbol{g}_y, \boldsymbol{h}_z)^2}{\mathcal{E}_0(\boldsymbol{f}_x)^2 \cdot \mathcal{E}_1(\boldsymbol{f}_x)^2} \cdot \prod_{q \mid N} c_q \cdot \frac{L(\boldsymbol{f}_x^\circ \otimes \boldsymbol{g}_y^\circ \otimes \boldsymbol{h}_z^\circ, c)}{\pi^{2(k-2)} \cdot \|\boldsymbol{f}_x^\circ\|^2},$$

where:

- $c = (k + \ell + m - 2)/2$,
- $\Gamma(k, \ell, m) = (c - 1)! \cdot (c - m)! \cdot (c - \ell)! \cdot (c + 1 - \ell - m)!$,
- $\alpha(k, \ell, m) \in \mathcal{R}$ *is a linear form in the variables* k, ℓ, m,
- $\mathcal{E}(\boldsymbol{f}_x, \boldsymbol{g}_y, \boldsymbol{h}_z) = (1 - \frac{\beta_{\boldsymbol{f}_x}\alpha_{\boldsymbol{g}_y}\alpha_{\boldsymbol{h}_z}}{p^c})(1 - \frac{\beta_{\boldsymbol{f}_x}\beta_{\boldsymbol{g}_y}\alpha_{\boldsymbol{h}_z}}{p^c})(1 - \frac{\beta_{\boldsymbol{f}_x}\alpha_{\boldsymbol{g}_y}\beta_{\boldsymbol{h}_z}}{p^c})(1 - \frac{\beta_{\boldsymbol{f}_x}\beta_{\boldsymbol{g}_y}\beta_{\boldsymbol{h}_z}}{p^c})$,
- $\mathcal{E}_0(\boldsymbol{f}_x) = (1 - \frac{\beta_{\boldsymbol{f}_x}}{\alpha_{\boldsymbol{f}_x}})$, $\mathcal{E}_1(\boldsymbol{f}_x) = (1 - \frac{\beta_{\boldsymbol{f}_x}}{p\alpha_{\boldsymbol{f}_x}})$,

and $\|\boldsymbol{f}_x^\circ\|^2$ *is the Petersson norm of* $\boldsymbol{f}_x^\circ$ *on* $\Gamma_0(N_{\boldsymbol{f}})$.

Proof. This is [Hsi21, Theorem A]. The construction of $\mathscr{L}_p^f(\breve{\underline{\boldsymbol{f}}}, \breve{\underline{\boldsymbol{g}}}, \breve{\underline{\boldsymbol{h}}})$ under hypotheses (CR), (ev), and (sq) is given in [Hsi21, §3.6] (where it is denoted $\mathscr{L}_{\boldsymbol{F}}^{\boldsymbol{f}}$), and the proof of its interpolation property assuming ($\Sigma^- = \emptyset$) is contained in [Hsi21, §7]. □

Remark 2.2. The definition of $\mathscr{L}_p^f(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})$ makes sense for any choice of test vectors $\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}}$), and even though in our applications we shall use the choice provided by Theorem 2.1, in the following we shall also consider other choices (see esp. Theorem 3.6).

2.3. **Triple tensor product of big Galois representations.** Let $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ be a triple of primitive Hida families with $\chi_{\boldsymbol{f}}\chi_{\boldsymbol{g}}\chi_{\boldsymbol{h}} = \omega^{2a}$ for some $a \in \mathbf{Z}$. For $\boldsymbol{\phi} \in \{\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}\}$, let $V_{\boldsymbol{\phi}}$ be the natural lattice in $(\mathrm{Frac}\ \mathbb{I}_{\boldsymbol{\phi}})^2$ realizing the Galois representation $\rho_{\boldsymbol{\phi}}$ in the étale cohomology of modular curves (see [Oht00]), and set

$$\mathbb{V}_{\boldsymbol{fgh}} := V_{\boldsymbol{f}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}.$$

This has rank 8 over $\mathcal{R}$, and by hypothesis its determinant can be written as $\det \mathbb{V}_{\boldsymbol{fgh}} = \mathcal{X}^2 \varepsilon_{\mathrm{cyc}}$ for a p-ramified Galois character $\mathcal{X}$ taking the value $(-1)^a$ at complex conjugation. Similarly as in [How07, Def. 2.1.3], we define the *critical twist*

$$\mathbb{V}_{\boldsymbol{fgh}}^{\dagger} := \mathbb{V}_{\boldsymbol{fgh}} \otimes \mathcal{X}^{-1}.$$

More generally, for any multiple N of $N_{\boldsymbol{\phi}}$ one can define Galois modules $V_{\boldsymbol{\phi}}(N)$ by working in tame level N; these split non-canonically into a finite direct sum of the $\mathbb{I}_{\boldsymbol{\phi}}$-adic representations $V_{\boldsymbol{\phi}}$ (see [DR17, §1.5.3]), and they define $\mathbb{V}_{\boldsymbol{fgh}}^{\dagger}(N)$ for any N divisible by $\mathrm{lcm}(N_{\boldsymbol{f}}, N_{\boldsymbol{g}}, N_{\boldsymbol{h}})$.

If f is a classical specialization of $\boldsymbol{f}$ with associated p-adic Galois representation V_f, we let $\mathbb{V}_{f,\boldsymbol{gh}}$ be the quotient of $\mathbb{V}_{\boldsymbol{fgh}}$ given by

$$\mathbb{V}_{f,\boldsymbol{gh}} := V_f \otimes_{\mathcal{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathbb{I}} V_{\boldsymbol{h}}.$$

Denote by $\mathbb{V}_{f,\boldsymbol{gh}}^{\dagger}$ the corresponding quotient of $\mathbb{V}_{\boldsymbol{fgh}}^{\dagger}$, and by $\mathbb{V}_{f,\boldsymbol{gh}}^{\dagger}(N)$ its level N counterpart.

2.4. **Theta elements and factorization.** We recall the factorization proven in [Hsi21, §8]. Let $f \in S_2(pN_f)$ be a p-stabilized newform of tame level N_f defined over $\mathcal{O}$, let $f^\circ \in S_2(N_f)$ be the associated newform, and let $\alpha_p = \alpha_p(f) \in \mathcal{O}^\times$ be the U_p-eigenvalue of f. Let K be an imaginary quadratic field of discriminant D_K prime to N_f. Write

$$N_f = N^+ N^-$$

with N^+ (resp. N^-) divisible only by primes which are split (resp. inert) in K, and choose an ideal $\mathfrak{N}^+ \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N}^+ \simeq \mathbf{Z}/N^+\mathbf{Z}$.

Assume that $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K, with our fixed embedding $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ inducing the prime $\mathfrak{p}$. Let Γ_∞ be the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension K_∞/K, fix a topological generator $\boldsymbol{\gamma} \in \Gamma_\infty$, and identity $\mathcal{O}[\![\Gamma_\infty]\!]$ with the power series ring $\mathcal{O}[\![T]\!]$ via $\boldsymbol{\gamma} \mapsto 1 + T$. For any prime-to-$p$ ideal $\mathfrak{a}$ of K, let $\sigma_{\mathfrak{a}}$ be the image of $\mathfrak{a}$ in the Galois group of the ray class field $K(p^\infty)/K$ of conductor p^∞ under the geometrically normalized reciprocity law map.

Theorem 2.3. *Let χ be a ring class character of K of conductor $c\mathcal{O}_K$ with values in $\mathcal{O}$, and assume that:*

(i) $(pN_f, cD_K) = 1$,
(ii) *N^- is the squarefree product of an odd number of primes,*
(iii) *if $q|N^-$ is a prime with $q \equiv 1 \pmod p$, then $\bar{\rho}_f$ is ramified at q.*

There exists a unique element $\Theta_{f/K,\chi}(T) \in \mathcal{O}[\![T]\!] \otimes_{\mathcal{O}} \mathrm{Frac}\,\mathcal{O}$ such that for every p-power root of unity ζ:

$$\Theta_{f/K,\chi}(\zeta - 1)^2 = \frac{p^n}{\alpha_p^{2n}} \cdot \mathcal{E}_p(f, \chi, \zeta)^2 \cdot \frac{L(f^\circ/K \otimes \chi\epsilon_\zeta, 1)}{(2\pi)^2 \cdot 4\|f^\circ\|^2_{\Gamma_0(N_{f^\circ})}} \cdot u_K^2 \sqrt{D_K} \chi\epsilon_\zeta(\sigma_{\mathfrak{N}^+}) \cdot \varepsilon_p,$$

where:

- $n \geqslant 0$ *is such that* ζ *has exact order* p^n,
- $\epsilon_\zeta : \Gamma_\infty \to \mu_{p^\infty}$ *be the character defined by* $\epsilon_\zeta(\gamma) = \zeta$,
- $\mathcal{E}_p(f, \chi, \zeta) = \begin{cases} (1-\alpha_p^{-1}\chi(\mathfrak{p}))(1-\alpha_p\chi(\overline{\mathfrak{p}})) & \text{if } n = 0, \\ 1 & \text{if } n > 0, \end{cases}$
- $\sigma_{\mathfrak{N}^+} \in \Gamma_\infty$ *is the image of* $\mathfrak{N}^+$ *under the geometrically normalized Artin's reciprocity map,*
- $u_K = |\mathcal{O}_K^\times|/2$, *and* $\varepsilon_p \in \{\pm 1\}$ *is the local root number of* f° *at* p.

Proof. See [BD96] for the first construction, and [CH18, Thm. A] for the stated interpolation property. □

When χ is the trivial character, we write $\Theta_{f/K,\chi}(T)$ simply as $\Theta_{f/K}(T)$. Suppose now that the newform f as in Theorem 2.3 is the specialization of a primitive Hida family $\boldsymbol{f} \in S^o(N_f, \mathbb{I})$ with branch character $\chi_{\boldsymbol{f}} = \mathbb{1}$ at an arithmetic point $x_1 \in \mathfrak{X}_{\mathbb{I}}^+$ of weight 2. Let $\ell \nmid pN_f$ be a prime split in K, and let χ be a ring class character of K of conductor $\ell^m\mathcal{O}_K$ for some $m > 0$. Suppose that $\chi = \psi^{1-\tau}$ with ψ a ray class character modulo $\ell^m\mathcal{O}_K$. Set $C = D_K\ell^{2m}$ and let

$$\boldsymbol{g} = \boldsymbol{\theta}_\psi(S_2) \in \mathcal{O}[\![S_2]\!][\![q]\!], \quad \boldsymbol{g}^* = \boldsymbol{\theta}_{\psi^{-1}}(S_3) \in \mathcal{O}[\![S_3]\!][\![q]\!]$$

be the primitive CM Hida families of level C constructed in [Hsi21, §8.3]. The p-adic triple product L-function of Theorem 2.1 attached to the triple $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{g}^*)$ (taking $a = -1$ in (ev)) is an element in $\mathcal{R} = \mathbb{I}[\![S_2, S_3]\!]$; in the following we let

$$\mathscr{L}_p^f(\underline{\breve{f}}, \underline{\breve{\boldsymbol{g}}\breve{\boldsymbol{g}}}^*) \in \mathcal{O}[\![S]\!]$$

denote the restriction to the "line" $S = S_2 = S_3$ of its image under the specialization map at x_1.

Let $\mathbb{K}_\infty$ be the $\mathbf{Z}_p^2$-extension of K, and let $K_{\mathfrak{p}^\infty}$ denote the $\mathfrak{p}$-ramified $\mathbf{Z}_p$-extension in $\mathbb{K}_\infty$, with Galois group $\Gamma_{\mathfrak{p}^\infty} = \mathrm{Gal}(K_{\mathfrak{p}^\infty}/K)$. Let $\gamma_\mathfrak{p} \in \Gamma_{\mathfrak{p}^\infty}$ be a topological generator, and for the formal variable T let $\Psi_T : \mathrm{Gal}(\mathbb{K}_\infty/K) \to \mathcal{O}[\![T]\!]^\times$ be the universal character defined by

$$\Psi_T(\sigma) = (1+T)^{l(\sigma)}, \quad \text{where } \sigma|_{K_{\mathfrak{p}^\infty}} = \gamma_\mathfrak{p}^{l(\sigma)}. \tag{2.3}$$

Denoting by the superscript τ the action of the non-trivial automorphism of $K/\mathbf{Q}$, the character $\Psi_T^{1-\tau}$ factors through Γ_∞ and yields an identification $\mathcal{O}[\![\Gamma_\infty]\!] \simeq \mathcal{O}[\![T]\!]$ corresponding to the topological generator $\gamma_\mathfrak{p}^{1-\tau} \in \Gamma_\infty$. Let p^b be the order of the p-part of the class number of K. Hereafter, we shall fix $\mathbf{v} \in \overline{\mathbf{Z}}_p^\times$ such that $\mathbf{v}^{p^b} = \varepsilon_{\mathrm{cyc}}(\gamma_\mathfrak{p}^{p^b}) \in 1 + p\mathbf{Z}_p$. Let $K(\chi, \alpha_p)/K$ (resp. $K(\chi)/K$) be the finite extension obtained by adjoining to K the values of χ and α_p (resp. the values of χ).

Proposition 2.4. *Set* $T = \mathbf{v}^{-1}(1+S) - 1$. *Then*

$$\mathscr{L}_p^f(\underline{\breve{f}}, \underline{\breve{\boldsymbol{g}}\breve{\boldsymbol{g}}}^*) = \pm\Psi_T^{\tau-1}(\sigma_{\mathfrak{N}^+}) \cdot \Theta_{f/K}(T) \cdot C_{f,\chi} \cdot \sqrt{L^{\mathrm{alg}}(f/K \otimes \chi, 1)},$$

where $C_{f,\chi} \in K(\chi, \alpha_p)^\times$ *and*

$$L^{\mathrm{alg}}(f/K \otimes \chi, 1) := \frac{L(f/K \otimes \chi, 1)}{4\pi^2\|f^\circ\|^2_{\Gamma_0(N_{f^\circ})}} \in K(\chi).$$

Proof. This is the factorization formula of [Hsi21, Prop. 8.1] specialized to $S = S_2 = S_3$, using the interpolation property of $\Theta_{f/K,\chi}(T)$ at $\zeta = 1$. □

Remark 2.5. The factorization of Proposition 2.4 reflects the decomposition of Galois representations

$$\mathbb{V}^\dagger_{f,\boldsymbol{g}\boldsymbol{g}^*} = \left(V_f(1) \otimes \mathrm{Ind}_K^{\mathbf{Q}}\Psi_T^{1-\tau}\right) \oplus \left(V_f(1) \otimes \mathrm{Ind}_K^{\mathbf{Q}}\chi\right). \tag{2.4}$$

Note that the first summand is the anticyclotomic deformation of $V_f(1)$, while the second is a fixed character twist of $V_f(1)$.

3. Coleman map for relative Lubin–Tate groups

In this section we review some elements of Perrin-Riou's theory [PR94] of big exponential maps, as extended by Kobayashi [Kob18] to $\mathbf{Z}_p$-extensions arising from relative Lubin–Tate groups of height one. Applied to local extensions arising from the anticyclotomic $\mathbf{Z}_p$-extension of an imaginary quadratic field K in which p splits, we deduce, by the results of §2 and [DR17], a Coleman power series construction of the p-adic L-function $\Theta_{f/K}$ of Theorem 2.3 that will play an important role later.

3.1. **Preliminaries.** Fix a complete algebraic closure $\mathbf{C}_p$ of $\mathbf{Q}_p$. Let $\mathbf{Q}_p^{\mathrm{ur}} \subset \mathbf{C}_p$ be the maximal unramified extension of $\mathbf{Q}_p$, and let $\mathrm{Fr} \in \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{ur}}/\mathbf{Q}_p)$ be the absolute Frobenius. Let $F \subset \mathbf{Q}_p^{\mathrm{ur}}$ be a finite unramified extension of $\mathbf{Q}_p$ with valuation ring $\mathscr{O}$ and set

$$R = \mathscr{O}[\![X]\!].$$

Let $\mathcal{F} = \mathrm{Spf}\, R$ be a relative Lubin–Tate formal group of height one defined over $\mathscr{O}$, and for each $n \in \mathbf{Z}$ set

$$\mathcal{F}^{(n)} := \mathcal{F} \times_{\mathrm{Spec}\,\mathscr{O}, \mathrm{Fr}^{-n}} \mathrm{Spec}\,\mathscr{O}.$$

The Frobenius morphism $\varphi_{\mathcal{F}} \in \mathrm{Hom}(\mathcal{F}, \mathcal{F}^{(-1)})$ induces a homomorphism $\varphi_{\mathcal{F}} \colon R \to R$ defined by

$$\varphi_{\mathcal{F}}(f) := f^{\mathrm{Fr}} \circ \varphi_{\mathcal{F}},$$

where f^{Fr} is the conjugate of f by Fr. Let $\psi_{\mathcal{F}}$ be the left inverse of $\varphi_{\mathcal{F}}$ satisfying

$$\varphi_{\mathcal{F}} \circ \psi_{\mathcal{F}}(f) = p^{-1} \sum_{x \in \mathcal{F}[p]} f(X \oplus_{\mathcal{F}} x). \tag{3.1}$$

Let F_∞/F be the Lubin–Tate $\mathbf{Z}_p^\times$-extension of F associated with $\mathcal{F}$, i.e., $F_\infty = \bigcup_{n=1}^\infty F(\mathcal{F}[p^n])$, and for every $n \geqslant -1$ let F_n be the subfield of F_∞ with $\mathrm{Gal}(F_n/F) \simeq (\mathbf{Z}/p^{n+1}\mathbf{Z})^\times$. (Hence, $F_{-1} = F$.) Letting $G_\infty = \mathrm{Gal}(F_\infty/F)$, there is a canonical decomposition

$$G_\infty \simeq \Delta \times \Gamma_\infty^{\mathcal{F}},$$

with Δ the torsion subgroup of G_∞ and $\Gamma_\infty^{\mathcal{F}} \simeq \mathbf{Z}_p$ the maximal torsion-free quotient of G_∞.

For every $a \in \mathbf{Z}_p^\times$, there is a unique formal power series $[a] \in R$ such that

$$[a]^{\mathrm{Fr}} \circ \varphi_{\mathcal{F}} = \varphi_{\mathcal{F}} \circ [a] \quad \text{and} \quad [a](X) \equiv aX \pmod{X^2}.$$

Letting $\varepsilon_{\mathcal{F}} \colon G_\infty \xrightarrow{\sim} \mathbf{Z}_p^\times$ be the Lubin–Tate character, we let $\sigma \in G_\infty$ act on $f \in R$ by

$$\sigma.f(X) := f([\varepsilon_{\mathcal{F}}(\sigma)](X)),$$

thus making R into an $\mathscr{O}[\![G_\infty]\!]$-module.

Lemma 3.1. *$R^{\psi_{\mathcal{F}}=0}$ is free of rank one over $\mathscr{O}[\![G_\infty]\!]$.*

Proof. This is a standard fact, see e.g. [Kob18, Prop. 5.4]. □

Let V be a crystalline $G_{\mathbf{Q}_p}$-representation defined over a finite extension L of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_L$. Let $\mathbf{D}(V) = \mathbf{D}_{\mathrm{cris},\mathbf{Q}_p}(V)$ be the filtered φ-module associated with V and set

$$\mathscr{D}_\infty(V) := \mathbf{D}(V) \otimes_{\mathbf{Z}_p} R^{\psi_{\mathcal{F}}=0}.$$

Fix an invariant differential $\omega_{\mathcal{F}} \in \Omega_R$, and let $\log_{\mathcal{F}} \in R\widehat{\otimes}\mathbf{Q}_p$ be the logarithm map satisfying

$$\log_{\mathcal{F}}(0) = 0 \quad \text{and} \quad d\log_{\mathcal{F}} = \omega_{\mathcal{F}},$$

where $d : R \to \Omega_R$ be the standard derivation.

Let $\epsilon = (\epsilon_n) \in T_p\mathcal{F} = \varprojlim \mathcal{F}^{(n)}[p^n]$ be a basis of the Tate module of $\mathcal{F}$, where the limit is with respect to the transition maps

$$\varphi^{\mathrm{Fr}^{-(n+1)}}\colon \mathcal{F}^{(n+1)}[p^{n+1}] \to \mathcal{F}^{(n)}[p^n].$$

One can associate to ϵ and $\omega_{\mathcal{F}}$ a p-adic period $t_\epsilon \in B^+_{\mathrm{cris}}$ such that

$$\mathbf{D}_{\mathrm{cris},F}(\varepsilon_{\mathcal{F}}) = Ft_\epsilon^{-1} \quad \text{and} \quad \varphi t_\epsilon = \varpi t_\epsilon, \tag{3.2}$$

where ϖ is the uniformizer in F such that $\varphi^*_{\mathcal{F}}(\omega^{\mathrm{Fr}}_{\mathcal{F}}) = \varpi \cdot \omega_{\mathcal{F}}$ (see [Kob18, §9.2]). For $j \in \mathbf{Z}$, the Lubin–Tate twist $V\langle j\rangle := V \otimes_L \varepsilon^j_{\mathcal{F}}$ then satisfies

$$\mathbf{D}_{\mathrm{cris},F}(V\langle j\rangle) = \mathbf{D}(V) \otimes_{\mathbf{Q}_p} Ft_\epsilon^{-j}.$$

There is a derivation $\mathrm{d}_\epsilon : \mathscr{D}_\infty(V\langle j\rangle) = \mathbf{D}_{\mathrm{cris},F}(V\langle j\rangle) \otimes_{\mathscr{O}} R^{\psi_{\mathcal{F}}=0} \to \mathscr{D}_\infty(V\langle j-1\rangle)$ given by

$$d_\epsilon : f = \eta \otimes g \mapsto \eta t_\epsilon \otimes \partial g,$$

where $\partial\colon R \to R$ is defined by $df = \partial f \cdot \omega_{\mathcal{F}}$. These give rise to the map

$$\widetilde{\Delta}\colon \mathscr{D}_\infty(V) \to \bigoplus_{j\in\mathbf{Z}} \frac{\mathbf{D}_{\mathrm{cris},F}(V\langle -j\rangle)}{1-\varphi} \tag{3.3}$$

sending $f \mapsto (\partial^j f(0) t_\epsilon^j \pmod{1-\varphi})_j$.

3.2. **Perrin-Riou's big exponential map.** For a finite extension K over $\mathbf{Q}_p$, let

$$\exp_{K,V}\colon \mathbf{D}(V) \otimes_{\mathbf{Q}_p} K \to \mathrm{H}^1(K,V)$$

be Bloch–Kato's exponential map [BK90, §3]. In this subsection, we recall the main properties of Perrin-Riou's map $\Omega_{V,h}$ interpolating $\exp_{K,V\langle j\rangle}$ over non-negative $j \in \mathbf{Z}$.

Let $V^* := \mathrm{Hom}_L(V, L(1))$ be the Kummer dual of V and denote by

$$[-,-]_V : \mathbf{D}(V^*) \otimes K \times \mathbf{D}(V) \otimes K \to L \otimes K$$

the K-linear extension of the de Rham pairing

$$\langle\, ,\, \rangle_{\mathrm{dR}}\colon \mathbf{D}(V^*) \times \mathbf{D}(V) \to L.$$

Let $\exp^*_{K,V} : \mathrm{H}^1(K,V) \to \mathbf{D}(V) \otimes K$ be the Bloch–Kato dual exponential map, which is characterized uniquely by

$$\mathrm{Tr}_{K/\mathbf{Q}_p}([x, \exp^*_{K,V}(y)]_V) = \langle \exp_{K,V^*}(x), y\rangle_{\mathrm{dR}},$$

for all $x \in \mathbf{D}(V^*) \otimes K$ and $y \in \mathrm{H}^1(K,V)$.

Choose a $\mathcal{O}_L$-lattice $T \subset V$ stable under the Galois action, and set $\widehat{\mathrm{H}}^1(F_\infty, T) = \varprojlim \mathrm{H}^1(F_n, T)$ and

$$\widehat{\mathrm{H}}^1(F_\infty, V) = \widehat{\mathrm{H}}^1(F_\infty, T) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,$$

which does not depend on the choice of T. Denote by

$$\mathrm{Tw}^j : \widehat{\mathrm{H}}^1(F_\infty, V) \simeq \widehat{\mathrm{H}}^1(F_\infty, V\langle j\rangle)$$

the twisting map by $\varepsilon^j_{\mathcal{F}}$. For a non-negative real number r, put

$$\mathscr{H}_{r,K}(X) = \left\{ \sum_{n\geqslant 0, \tau\in\Delta} c_{n,\tau} \cdot \tau \cdot X^n \in K[\Delta][\![X]\!] \mid \sup_n |c_{n,\tau}|_p\, n^{-r} < \infty \text{ for all } \tau \in \Delta \right\},$$

where $|\cdot|_p$ is the normalized valuation of K with $|p|_p = p^{-1}$. Let $\boldsymbol{\gamma}$ be a topological generator of $\Gamma^{\mathcal{F}}_\infty$, and denote by $\mathscr{H}_{r,K}(G_\infty)$ the ring of elements $\{f(\boldsymbol{\gamma}-1)\colon f \in \mathscr{H}_{r,K}(X)\}$, so in particular $\mathscr{H}_{0,K}(G_\infty) = \mathcal{O}_K[\![G_\infty]\!] \otimes_{\mathcal{O}_K} K$. Put

$$\mathscr{H}_{\infty,K}(G_\infty) = \bigcup_{r\geqslant 0} \mathscr{H}_{r,K}(G_\infty).$$

Define the map

$$\Xi_{n,V} \colon \mathbf{D}(V) \otimes_{\mathbf{Q}_p} \mathscr{H}_{\infty,F}(X) \to \mathbf{D}(V) \otimes_{\mathbf{Q}_p} F_n$$

by

$$\Xi_{n,V}(G) := \begin{cases} p^{-(n+1)}\varphi^{-(n+1)}(G^{\mathrm{Fr}^{-(n+1)}}(\epsilon_n)) & \text{if } n \geqslant 0, \\ (1-p^{-1}\varphi^{-1})(G(0)) & \text{if } n = -1, \end{cases} \tag{3.4}$$

and let $\widetilde{\Lambda} = \mathbf{Z}_p[\![G_\infty]\!]$.

Theorem 3.2. *Let $\epsilon = (\epsilon_n)$ be a basis of $T_p\mathcal{F}$, let $h > 0$ be such that $\mathbf{D}(V) = \mathrm{Fil}^{-h}\,\mathbf{D}(V)$, and assume that $\mathrm{H}^0(F_\infty, V) = 0$. There exists $\widetilde{\Lambda}$-linear "big exponential map"*

$$\Omega^\epsilon_{V,h} : \mathscr{D}_\infty(V)^{\widetilde{\Delta}=0} \to \widehat{\mathrm{H}}^1(F_\infty, T) \otimes_{\widetilde{\Lambda}} \mathscr{H}_{\infty,F}(G_\infty)$$

such that for every $g \in \mathscr{D}_\infty(V)^{\widetilde{\Delta}=0}$ and $j \geqslant 1-h$ satisfies the interpolation property

$$\mathrm{pr}_{F_n}(\mathrm{Tw}^j \circ \Omega^\epsilon_{V,h}(g)) = (-1)^{h+j-1}(h+j-1)! \cdot \exp_{F_n,V\langle j\rangle}(\Xi_{n,V\langle j\rangle}(\mathrm{d}_\epsilon^{-j}G)) \in \mathrm{H}^1(F_n, V\langle j\rangle),$$

where $G \in \mathbf{D}(V) \otimes_{\mathbf{Q}_p} \mathscr{H}_{h,F}(X)$ is a solution of the equation

$$(1 - \varphi \otimes \varphi_{\mathcal{F}})G = g.$$

Moreover, these maps satisfy

$$\mathrm{Tw}^j \circ \Omega^\epsilon_{V,h} \circ \mathrm{d}^j_\epsilon = \Omega^\epsilon_{V\langle j\rangle,h+j},$$

and if $j \leqslant -h$ then

$$\exp^*_{F_n,V\langle j\rangle}(\mathrm{pr}_{F_n}(\mathrm{Tw}_j \circ \Omega^\epsilon_{V,h}(g))) = \frac{1}{(-h-j)!} \cdot \Xi_{n,V\langle j\rangle}(\mathrm{d}_\epsilon^{-j}G)) \in \mathbf{D}(V\langle j\rangle) \otimes_{\mathbf{Q}_p} F_n;$$

and if $D_{[s]} \subset \mathbf{D}(V)$ is a φ-invariant subspace in which all φ-eigenvalues have p-adic valuation at most s, then $\Omega^\epsilon_{V,h}$ maps $(D_{[s]} \otimes_{\mathbf{Z}_p} R^{\psi_{\mathcal{F}}=0})^{\widetilde{\Delta}=0}$ into $\widehat{\mathrm{H}}^1(F_\infty, T) \otimes_{\widetilde{\Lambda}} \mathscr{H}_{s+h,F}(G_\infty)$.

Proof. For $\mathcal{F} = \widehat{\mathbf{G}}_m$, the construction of $\Omega^\epsilon_{V,h}$ and its interpolation property for $j \geqslant 1-h$ is due to Perrin-Riou [PR94]; the interpolation formula for $j \leqslant -h$ is due to Colmez [Col98]. The extension of these results to $\mathbf{Z}_p$-extensions arising from relative Lubin–Tate formal groups of height one is given in [Kob18, Appendix]. ☐

3.3. **The Coleman map.** From now on, we assume that

$$\mathscr{D}_\infty(V)^{\widetilde{\Delta}=0} = \mathscr{D}_\infty(V), \tag{3.5}$$

i.e., $\widetilde{\Delta} = 0$ (note that by (3.3) this is a condition on the φ-eigenvalues on $\mathbf{D}_{\mathrm{cris},F}(V)$), and for simplicity for any field extension $M/\mathbf{Q}_p$ we write $\mathscr{H}_M$ for $\mathscr{H}_{0,M}(G_\infty)$. Let

$$[-,-]_V : \mathbf{D}(V^*) \otimes_{\mathbf{Q}_p} \mathscr{H}_F \times \mathbf{D}(V) \otimes_{\mathbf{Q}_p} \mathscr{H}_F \to L \otimes_{\mathbf{Q}_p} \mathscr{H}_F$$

be the pairing defined by

$$[\eta_1 \otimes \lambda_1, \eta_2 \otimes \lambda_2]_V = \langle \eta_1, \eta_2\rangle_{\mathrm{dR}} \otimes \lambda_1\lambda_2^\iota$$

for all $\lambda_1, \lambda_2 \in \mathscr{H}_F$.

Recall that $F_\infty = \bigcup_n F_n$, and let $\langle -,-\rangle_{F_n}$ be the local Tate pairing $\mathrm{H}^1(F_n, T^*) \times \mathrm{H}^1(F_n, T) \to \mathcal{O}_L$. Letting $x = (x_n)_n$ and $y = (y_n)_n$ be sequences in $\widehat{\mathrm{H}}^1(F_\infty, T^*)$ and $\widehat{\mathrm{H}}^1(F_\infty, T)$, these extend to a $\mathcal{O}_L[\![G_\infty]\!]$-linear pairing

$$\langle -,-\rangle_{F_\infty} : \widehat{\mathrm{H}}^1(F_\infty, T^*) \times \widehat{\mathrm{H}}^1(F_\infty, T) \to \mathcal{O}_L[\![G_\infty]\!]$$

by defining $\langle x, y\rangle_{F_\infty}$ to be the limit of the compatible elements $\sum_{\sigma \in \mathrm{Gal}(F_n/F)} \langle x_n^{\sigma^{-1}}, y_n\rangle_{F_n}[\sigma] \in \mathcal{O}_L[\mathrm{Gal}(F_n/F)]$. After inverting p, this extends to a pairing

$$\langle -,-\rangle_{F_\infty} : \widehat{\mathrm{H}}^1(F_\infty, V^*) \times \widehat{\mathrm{H}}^1(F_\infty, V) \to L \otimes_{\mathbf{Q}_p} \mathscr{H}_{\mathbf{Q}_p}. \tag{3.6}$$

Definition 3.3. Let $\boldsymbol{e} \in R^{\psi_{\mathcal{F}}=0}$ be a $\mathscr{O}[\![G_\infty]\!]$-module generator, and let ϵ a generator of $T_p\mathcal{F}$. The *Coleman map*

$$\mathrm{Col}^{\epsilon}_{\boldsymbol{e}} \colon \widehat{\mathrm{H}}^1(F_\infty, V^*) \to \mathbf{D}(V^*) \otimes_{\mathbf{Q}_p} \mathscr{H}_F$$

is the $L \otimes_{\mathbf{Q}_p} \mathscr{H}_F$-linear map uniquely characterized by

$$\mathrm{Tr}_{F/\mathbf{Q}_p}([\mathrm{Col}^{\epsilon}_{\boldsymbol{e}}(\mathbf{z}), \eta]_V) = \langle \mathbf{z}, \Omega^{\epsilon}_{V,h}(\eta \otimes \boldsymbol{e}) \rangle_{F_\infty} \tag{3.7}$$

for all $\eta \in \mathbf{D}(V)$.

Let $\mathcal{Q}$ be the completion of $\mathbf{Q}_p^{\mathrm{ur}}$ in $\mathbf{C}_p$, with ring of integers $\mathcal{W}$, and set $F_n^{\mathrm{ur}} = F_n\mathbf{Q}_p^{\mathrm{ur}}$ for $-1 \leqslant n \leqslant \infty$ (so $F_{-1}^{\mathrm{ur}} = F^{\mathrm{ur}}$). Let $\sigma_0 \in \mathrm{Gal}(F_\infty^{\mathrm{ur}}/\mathbf{Q}_p)$ be such that $\sigma_0|_{\mathbf{Q}_p^{\mathrm{ur}}} = \mathrm{Fr}$ is the absolute Frobenius.

Fix an isomorphism

$$\rho : \widehat{\mathbf{G}}_m \simeq \mathcal{F} \tag{3.8}$$

defined over $\mathcal{W}$ and let $\rho : \mathcal{W}[\![T]\!] \simeq R \otimes_{\mathscr{O}} \mathcal{W}$ be the map defined by $\rho(f) = f \circ \rho^{-1}$, so

$$\varphi_{\mathcal{F}} \circ \rho = \rho^{\mathrm{Fr}} \circ \varphi_{\widehat{\mathbf{G}}_m}.$$

Fix also a $\mathscr{O}[\![G_\infty]\!]$-generator $\boldsymbol{e} \in R^{\psi_{\mathcal{F}}=0}$, and let $h_{\boldsymbol{e}} \in \mathcal{W}[\![G_\infty]\!]$ be such that $\rho(1+X) = h_{\boldsymbol{e}} \cdot \boldsymbol{e}$. Note that $\boldsymbol{e}(0) \in \mathscr{O}^\times$. Fix a sequence (ζ_{p^n}) of primitive p^n-th root of unity giving a generator of $T_p\widehat{\mathbf{G}}_m$, and let $\epsilon = (\epsilon_n)$ be the generator of $T_p\mathcal{F}$ given by

$$\epsilon_n = \rho^{\mathrm{Fr}^{-(n+1)}}(\zeta_{p^{n+1}} - 1) \in \mathcal{F}^{(n+1)}[p^{n+1}].$$

Let $t \in B^+_{\mathrm{cris}}$ be the p-adic period as in §3.1 associated to the generator $(\zeta_{p^{n+1}} - 1) \in T_p\widehat{\mathbf{G}}_m$ and the invariant differential $\omega_{\widehat{\mathbf{G}}_m} = \frac{dX}{1+X}$.

From now on, we suppose that $\mathrm{Fil}^{-1}\mathbf{D}(V) = \mathbf{D}(V)$ and $H^0(F_\infty, V) = 0$, so the big exponential map $\Omega^{\epsilon}_{V,1}$ of Theorem 3.2 is defined. Let $\eta \in \mathbf{D}(V)$ be such that $\varphi\eta = \alpha\eta$, and suppose that η has slope s (i.e. $|\alpha|_p = p^{-s}$). For every $\mathbf{z} \in \widehat{\mathrm{H}}^1(F_\infty, V^*)$, we define

$$\mathrm{Col}^{\eta}(\mathbf{z}) := \sum_{j=1}^{[F:\mathbf{Q}_p]} \left[\mathrm{Col}^{\epsilon}_{\boldsymbol{e}}(\mathbf{z}^{\sigma_0^{-j}}), \eta\right] \cdot h_{\boldsymbol{e}} \cdot \sigma_0^j \in \mathscr{H}_{s+h, L\mathcal{Q}}(\widetilde{G}_\infty), \tag{3.9}$$

where $\widetilde{G}_\infty = \mathrm{Gal}(F_\infty/\mathbf{Q}_p)$, and $[-,-] : \mathbf{D}(V^*) \otimes \mathscr{H}_{\mathcal{Q}} \times \mathbf{D}(V) \otimes \mathscr{H}_{\mathcal{Q}} \to \mathscr{H}_{L\mathcal{Q}}$ is the image of $[-,-]_V$ under the natural map $L \otimes_{\mathbf{Q}_p} \mathscr{H}_{\mathcal{Q}} \to \mathscr{H}_{L\mathcal{Q}}$. We put

$$\mathbf{z}_{-j,n} := \mathrm{pr}_{F_n}(\mathrm{Tw}^{-j}(\mathbf{z})) \in \mathrm{H}^1(F_n, V^*\langle -j\rangle),$$

and say that a finite order character χ of $\widetilde{G}_\infty$ has conductor p^{n+1} if n is the smallest integer such that χ factors through $\mathrm{Gal}(F_n/\mathbf{Q}_p)$.

Theorem 3.4. *Let $\mathbf{z} \in \widehat{\mathrm{H}}^1(F_\infty, V^*)$ and let ψ be a p-adic character of $\widetilde{G}_\infty$ such that $\psi = \chi\varepsilon_{\mathcal{F}}^j$ with χ a finite order character of conductor p^{n+1}. If $j < 0$, then*

$$\mathrm{Col}^{\eta}(\mathbf{z})(\psi) = \frac{(-1)^{j-1}}{(-j-1)!} \times \begin{cases} \left[\log_{F,V^*\langle -j\rangle}(\mathbf{z}_{-j,n}) \otimes t^{-j}, (1-p^{j-1}\varphi^{-1})(1-p^{-j}\varphi)^{-1}\eta\right] & \textit{if } n = -1, \\ p^{(n+1)(j-1)}\boldsymbol{\tau}(\psi) \displaystyle\sum_{\tau \in \mathrm{Gal}(F_n/\mathbf{Q}_p)} \chi^{-1}(\tau) \left[\log_{F_n,V^*\langle -j\rangle}(\mathbf{z}^{\tau}_{-j,n}) \otimes t^{-j}, \varphi^{-(n+1)}\eta\right] & \textit{if } n \geqslant 0. \end{cases}$$

If $j \geqslant 0$, then

$$\mathrm{Col}^{\eta}(\mathbf{z})(\psi) = j!(-1)^j \times \begin{cases} \Big[\exp^*_{F,V^*\langle -j\rangle}(\mathbf{z}_{-j,n}) \otimes t^{-j}, (1-p^{j-1}\varphi^{-1})(1-p^{-j}\varphi)^{-1}\eta\Big] & \text{if } n=-1, \\ p^{(n+1)(j-1)}\boldsymbol{\tau}(\psi) \displaystyle\sum_{\tau \in \mathrm{Gal}(F_n/\mathbf{Q}_p)} \chi^{-1}(\tau)\Big[\exp^*_{F_n,V^*\langle -j\rangle}(\mathbf{z}^{\tau}_{-j,n}) \otimes t^{-j}, \varphi^{-(n+1)}\eta\Big] & \text{if } n \geqslant 0. \end{cases}$$

Here $\boldsymbol{\tau}(\psi)$ is the Gauss sum defined by

$$\boldsymbol{\tau}(\psi) := \sum_{\tau \in \mathrm{Gal}(F_n^{\mathrm{ur}}/F^{\mathrm{ur}})} \psi\varepsilon_{\mathrm{cyc}}^{-j}(\tau\sigma_0^{n+1})\zeta_{p^{n+1}}^{\tau\sigma_0^{n+1}}.$$

Proof. This follows from Theorem 3.2 by a direct computation (see [Kob18, Thm. 5.10], and [LZ14, Thm. 4.15] for a related computation). □

3.4. **Diagonal cycles and theta elements.** We now apply the local results of the preceding section to the global setting of §2. Assume that f, $\boldsymbol{g} = \boldsymbol{\theta}_{\psi}(S)$ and $\boldsymbol{g}^* = \boldsymbol{\theta}_{\psi^{-1}}(S)$ are as in §2.4. Keeping the notations from §2.3, by [DR17, §1] there exists a class

$$\kappa(f, \boldsymbol{g}\boldsymbol{g}^*) \in \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^{\dagger}_{f,\boldsymbol{g}\boldsymbol{g}^*}(N)) \tag{3.10}$$

constructed from twisted diagonal cycles on the triple product of modular curves of tame level N. (See also [DR21] and [BSV21].)

Every triple of test vectors $\breve{\boldsymbol{F}} = (\breve{f}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{g}}^*)$ defines a $G_{\mathbf{Q}}$-equivariant projection $\mathbb{V}^{\dagger}_{f,\boldsymbol{g}\boldsymbol{g}^*}(N) \to \mathbb{V}^{\dagger}_{f,\boldsymbol{g}\boldsymbol{g}^*}$, and we put

$$\kappa(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*) := \mathrm{pr}_{\breve{\boldsymbol{F}}}(\kappa(f, \boldsymbol{g}\boldsymbol{g}^*)) \in \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^{\dagger}_{f,\boldsymbol{g}\boldsymbol{g}^*}), \tag{3.11}$$

where $\mathrm{pr}_{\breve{\boldsymbol{F}}} : \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^{\dagger}_{f,\boldsymbol{g}\boldsymbol{g}^*}(N)) \to \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^{\dagger}_{f,\boldsymbol{g}\boldsymbol{g}^*})$ is the induced map on cohomology.

Since $\Psi_T^{1-\tau}$ gives the universal character of $\mathrm{Gal}(K_\infty/K)$, by the $G_{\mathbf{Q}}$-isomorphism (2.4) and Shapiro's lemma we have the identifications

$$\begin{aligned} \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^{\dagger}_{f,\boldsymbol{g}\boldsymbol{g}^*}) &\simeq \mathrm{H}^1(\mathbf{Q}, V_f(1) \otimes \mathrm{Ind}_K^{\mathbf{Q}} \Psi_T^{1-\tau}) \oplus \mathrm{H}^1(\mathbf{Q}, V_f(1) \otimes \mathrm{Ind}_K^{\mathbf{Q}} \chi) \\ &\simeq \widehat{\mathrm{H}}^1(K_\infty, V_f(1)) \oplus \mathrm{H}^1(K, V_f(1) \otimes \chi). \end{aligned} \tag{3.12}$$

In the following, we write

$$\kappa(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*) = (\kappa_\infty(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*), \kappa_0(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*)) \tag{3.13}$$

according to this decomposition.

Let g and g^* be the weight 1 eigenform θ_ψ and $\theta_{\psi^{-1}}$, respectively, so that the specialization of $(\boldsymbol{g}, \boldsymbol{g}^*)$ at $T = 0$ (or equivalently, $S = \mathbf{v} - 1$) is a p-stabilization of the pair (g, g^*).

Lemma 3.5. *Assume that $L(f \otimes g \otimes g^*, 1) = 0$ and that $L(f/K \otimes \chi, 1) \neq 0$. Then for every choice of test vectors $\breve{\boldsymbol{F}} = (\breve{f}, \breve{\boldsymbol{g}}, \boldsymbol{g}^*)$ we have $\kappa_0(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*) = 0$.*

Proof. Let $\boldsymbol{\kappa} = \kappa(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*)$ and for every $? \in \{f, \boldsymbol{g}, \boldsymbol{g}^*\}$, let $\mathscr{F}^+V_?$ be the rank one subspace of $V_?$ fixed by the inertia group at p. By (3.12), in order to prove (1) it suffices to show that some specialization of κ has trivial image in $\mathrm{H}^1(K, V_f(1) \otimes \chi)$. Let

$$\kappa_{\breve{f},\breve{g}\breve{g}^*} := \boldsymbol{\kappa}|_{S=\mathbf{v}-1} \in \mathrm{H}^1(\mathbf{Q}, V_{fgg^*}) = \mathrm{H}^1(K, V_f(1)) \oplus \mathrm{H}^1(K, V_f(1) \otimes \chi),$$

where $V_{fgg^*} := V_f(1) \otimes V_g \otimes V_{g^*}$. By considering Hodge–Tate weights, it is easily seen that the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V_{fgg^*}) \subset \mathrm{H}^1(\mathbf{Q}, V_{fgg^*})$ is given by

$$\mathrm{Sel}(\mathbf{Q}, V_{fgh}) = \ker\Big(\mathrm{H}^1(\mathbf{Q}, V_{fgg^*}) \overset{\partial_p \circ \mathrm{loc}_p}{\to} \mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}^- V_f(1) \otimes V_g \otimes V_{g^*})\Big),$$

where ∂_p is the natural map induced by the projection $V_f \twoheadrightarrow \mathscr{F}^- V_f := V_f/\mathscr{F}^+ V_f$ (see e.g. [DR17, p. 634]). Thus it follows that

$$\mathrm{Sel}(\mathbf{Q}, V_{fgg^*}) = \mathrm{Sel}(K, V_f(1)) \oplus \mathrm{Sel}(K, V_f(1) \otimes \chi).$$

The implications $L(f \otimes g \otimes g^*, 1) = 0 \Longrightarrow \kappa_{\breve{f},\breve{g}\breve{g}^*} \in \mathrm{Sel}(\mathbf{Q}, V_{fgg^*})$ and $L(f/K \otimes \chi, 1) \neq 0 \Longrightarrow \mathrm{Sel}(K, V_f(1) \otimes \chi) = 0$, which follow from [DR17, Thm. C] and [CH15, Thm. 1], respectively, therefore yield the result. ☐

Suppose from now on that $f^\circ \in S_2(N_f)$ is the newform associated to an elliptic curve $E/\mathbf{Q}$ with good ordinary reduction at p. Thus $V_f(1) \simeq V_pE$ and from (3.13) we obtain an Iwasawa cohomology class

$$\kappa_\infty(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*) \in \widehat{\mathrm{H}}^1(K_\infty, V_pE).$$

Set $V = V_pE$ for the ease of notation. Note that $\mathrm{Fil}^{-1}\mathbf{D}(V) = \mathbf{D}(V)$ and, by the Weil pairing, $V^* \simeq V$. Let $\mathfrak{P}$ be the prime of $\overline{\mathbf{Q}}$ above p induced by our fixed embedding ι_p (inducing $\mathfrak{p}$ on K), and for any subfield $H \subset \overline{\mathbf{Q}}$ denote by $\hat{H} = H_{\mathfrak{P}}$ the completion of H with respect to $\mathfrak{P}$. Then $\mathrm{Gal}(\hat{K}_\infty/\mathbf{Q}_p)$ is identified with the decomposition group of $\mathfrak{P}$ in $\Gamma_\infty = \mathrm{Gal}(K_\infty/K)$

Let H_c be the ring class field of K of conductor c, and put $F = \hat{H}_c$ for a fixed c prime to p. Let $\varpi \in K$ be a generator of $\mathfrak{p}^{[F:\mathbf{Q}_p]}$ and let F_∞/F be the Lubin–Tate $\mathbf{Z}_p$-extension associated with the uniformizer $\varpi/\overline{\varpi} \in \mathcal{O}_F$ (see [Kob18, §3.1]). As is well-known, we have

$$F_\infty = \bigcup_{n=0}^{\infty} \hat{H}_{cp^n}$$

(see e.g. [Shn16, Prop. 8.3]). In particular, F_∞ contains $\hat{K}_\infty$.

Let ω_E be the Néron differential of E, regarded as an element in $\mathbf{D}(\mathrm{H}^1_{\mathrm{et}}(E_{/\overline{\mathbf{Q}}}, \mathbf{Q}_p)) \simeq \mathbf{D}(V^*)$. Let $\alpha_p \in \mathbf{Z}_p^\times$ be the p-adic unit eigenvalue of the Frobenius map φ acting on $\mathbf{D}(V)$, and let $\eta \in \mathbf{D}(V) \simeq \mathbf{D}(\mathrm{H}^1_{\mathrm{et}}(E_{/\overline{\mathbf{Q}}}, \mathbf{Q}_p)) \otimes \mathbf{D}(\mathbf{Q}_p(1))$ be a φ-eigenvector of slope -1 such that

$$\varphi\eta = p^{-1}\alpha_p \cdot \eta \quad \text{and} \quad \langle \eta, \omega_E \otimes t^{-1} \rangle_{\mathrm{dR}} = 1. \tag{3.14}$$

Finally, note that hypothesis (3.5) holds since $\mathbf{D}(V)^{\varphi^{[F:\mathbf{Q}_p]} = (\varpi/\overline{\varpi})^j} = 0$ for any $j \in \mathbf{Z}$, given that the φ-eigenvalues of $\mathbf{D}(V)$ are p-Weil numbers while $\varpi/\overline{\varpi}$ is a 1-Weil number.

The second part of the next result recasts the “explicit reciprocity law” of [DR17, Thm. 5.3] (see also [DR21, Thm. 5.1] and [BSV21, Thm. A]) in terms of the Coleman map of §3.3.

Theorem 3.6. *Assume that $L(f \otimes g \otimes g^*, 1) = 0$ and that $L(f/K \otimes \chi, 1) \neq 0$. Then, for any test vectors $(\breve{f}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{g}}^*)$, we have*

$$\mathrm{Loc}_{\overline{\mathfrak{p}}}(\kappa_\infty(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*)) = 0,$$

and

$$\mathrm{Col}^\eta(\mathrm{Loc}_{\mathfrak{p}}(\kappa_\infty(\breve{f}, \breve{g}\breve{g}^*))) = \mathscr{L}_p^f(\breve{f}, \breve{g}\breve{g}^*) \cdot 2\alpha_p^{-1}(1 - \alpha_p^{-1}\chi(\overline{\mathfrak{p}}))^{-1}.$$

Proof. Let $\mathscr{F}^{++}\mathbb{V}^\dagger_{\boldsymbol{fgg}^*}$ be the rank four $G_{\mathbf{Q}_p}$-stable submodule of $\mathbb{V}^\dagger_{\boldsymbol{fgg}^*}$ defined by

$$\big[\mathscr{F}^+V \otimes \mathscr{F}^+V_{\boldsymbol{g}} \otimes V_{\boldsymbol{g}^*} + \mathscr{F}^+V \otimes V_{\boldsymbol{g}} \otimes \mathscr{F}^+V_{\boldsymbol{g}^*} + V \otimes \mathscr{F}^+V_{\boldsymbol{g}} \otimes \mathscr{F}^+V_{\boldsymbol{g}^*}\big] \otimes \mathcal{X}^{-1},$$

The class $\kappa(\breve{f}, \breve{g}\breve{g}^*) = (\kappa_\infty(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*), \kappa_0(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*)) \in \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{\boldsymbol{fgg}^*})$ is known to land in the kernel of the composite map

$$\mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{\boldsymbol{fgg}^*}) \xrightarrow{\mathrm{Loc}_p} \mathrm{H}^1(\mathbf{Q}_p, \mathbb{V}^\dagger_{\boldsymbol{fgg}^*}) \to \mathrm{H}^1(\mathbf{Q}_p, \mathbb{V}^\dagger_{\boldsymbol{fgg}^*}/\mathscr{F}^{++}\mathbb{V}^\dagger_{\boldsymbol{fgg}^*})$$

(see e.g. [DR21, Prop. 5.8]). Using (2.4), we immediately find that

$$\mathscr{F}^{++}\mathbb{V}^\dagger_{\boldsymbol{fgg}^*} = V \otimes \Psi_T^{1-\tau} + \mathscr{F}^+V \otimes (\chi + \chi^{-1}),$$

and therefore, identifying $G_{\mathbf{Q}_p}$ with $G_{K_\mathfrak{p}}$ via our fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, we obtain

$$\mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}^{++}\mathbb{V}^\dagger_{\boldsymbol{f}\boldsymbol{g}\boldsymbol{g}^*}) \simeq \mathrm{H}^1(K_\mathfrak{p}, V \otimes \Psi_T^{1-\tau}) \oplus \mathrm{H}^1(K_\mathfrak{p}, \mathscr{F}^+V \otimes \chi) \oplus \mathrm{H}^1(K_{\overline{\mathfrak{p}}}, \mathscr{F}^+V \otimes \chi).$$

This shows the vanishing of $\mathrm{Loc}_{\overline{\mathfrak{p}}}(\kappa_\infty(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*))$, and the second equality in the theorem follows from Lemma 3.5 and [DR17, Thm. 5.3]. □

Corollary 3.7. *Assume that $L(f \otimes g \otimes g^*, 1) = 0$ and that $L(f/K, \chi, 1) \neq 0$. Let $(\underline{\breve{f}}, \underline{\breve{\boldsymbol{g}}}, \underline{\breve{\boldsymbol{g}}}^*)$ be the triple of test vectors from Theorem 2.1. Then*

$$\mathrm{Loc}_{\overline{\mathfrak{p}}}(\kappa_\infty(\underline{\breve{f}}, \underline{\breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*})) = 0,$$

and

$$\mathrm{Col}^\eta(\mathrm{Loc}_\mathfrak{p}(\kappa_\infty(\underline{\breve{f}}, \underline{\breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*}))) = \pm\Psi_T^{\tau-1}(\sigma_{\mathfrak{N}^+}) \cdot \Theta_{f/K}(T) \cdot \sqrt{L^{\mathrm{alg}}(f/K \otimes \chi, 1)} \cdot C_{f,\chi} \frac{2C_{f,\chi}}{\alpha_p(1 - \alpha_p^{-1}\chi(\overline{\mathfrak{p}}))},$$

where $C_{f,\chi} \in K(\chi, \alpha_p)^\times$ is the non-zero algebraic number.

Proof. This is the combination of Theorem 3.6 and the factorization in Proposition 2.4. □

4. Anticyclotomic derived p-adic heights

The goal of this section is Theorem 4.5, giving a formula for the anticyclotomic derived p-adic heights in terms of the Coleman map introduced before. This formula is a generalization of Rubin's height formula [Rub94] in arbitrary rank.

4.1. The general theory.

Initiated in [BD95] and further developed in [How04], the theory of derived p-adic heights relates the degeneracies of the p-adic height to the failure of the p^∞-Selmer group of elliptic curves over a $\mathbf{Z}_p$-extension to be semi-simple as an Iwasawa module. Derived p-adic heights seem to have been rarely used for arithmetic applications in the previous literature[1], but they will play a key role in the proof of our results.

In this section we briefly recall the results from [How04] (with a slight generalization) that we will need.

Let E be an elliptic curve over $\mathbf{Q}$ of conductor N with good ordinary reduction at $p > 2$. For any number field F, let $\mathrm{Sel}_{p^r}(E/F) \subset \mathrm{H}^1(F, E[p^r])$ be the p^r-Selmer group of E over F, and put

$$\mathrm{Sel}(F, T_pE) = \varprojlim_r \mathrm{Sel}_{p^r}(E/F)$$

and $\mathrm{Sel}(F, V_pE) = \mathrm{Sel}(F, T_pE) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Let K be an imaginary quadratic field of discriminant prime to Np, and let K_∞/K be the anticyclotomic $\mathbf{Z}_p$-extension of K. Denote by K_n the subsection of K_∞ with $[K_n\colon K] = p^n$, and put

$$\mathrm{Sel}_{p^\infty}(E/K_\infty) = \varinjlim_n \mathrm{Sel}_{p^\infty}(E/K_n).$$

Finally, let $\Lambda = \mathbf{Z}_p[\![\mathrm{Gal}(K_\infty/K)]\!]$ be the anticyclotomic Iwasawa algebra, and denote by $J \subset \Lambda$ the augmentation ideal.

Theorem 4.1. *Let N^- be the largest factor of N divisible only by primes that are inert in K, and suppose that*

- *N^- is squarefree,*
- *$E[p]$ is ramified at every prime $q|N^-$.*

[1] Perhaps by influence of *cyclotomic* Iwasawa theory, a context in which the p-adic height is conjectured to be non-degenerate, see [Sch85].

Then there is a filtration

$$\mathrm{Sel}(K, V_pE) = S_p^{(1)}(E/K) \supset \cdots \supset S_p^{(i)}(E/K) \supset S_p^{(i+1)}(E/K) \supset \cdots \supset S_p^{(\infty)}(E/K)$$

and a sequence of height pairings

$$h_p^{(i)} : S_p^{(i)}(E/K) \times S_p^{(i)}(E/K) \to (J^i/J^{i+1}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

with the following properties:

(a) $S_p^{(i+1)}(E/K)$ *is the null-space of* $h_p^{(r)}$.

(b) $S_p^{(\infty)}(E/K)$ *is the subspace of* $\mathrm{Sel}(K, V_pE)$ *consisting of universal norms for* K_∞/K*:*

$$S_p^{(\infty)}(E/K) = \bigcap_{n=1}^{\infty} \mathrm{cor}_{K_n/K}(\mathrm{Sel}(K_n, V_pE)).$$

(c) $h_p^{(i)}$ *is symmetric (resp. alternating) for i odd (resp. i even).*

(d) $h_p^{(i)}(x^\tau, y^\tau) = (-1)^i h_p^{(i)}(x, y)$*, where* $\tau \in \mathrm{Gal}(K/\mathbf{Q})$ *is complex conjugation.*

(e) *Let*

$$e_i := \begin{cases} \dim_{\mathbf{Q}_p}(S_p^{(i)}(E/K)/S_p^{(i+1)}(E/K)) & \text{if } i < \infty, \\ \dim_{\mathbf{Q}_p} S_p^{(\infty)}(E/K) & \text{if } i = \infty. \end{cases}$$

Then there is a Λ*-module pseudo-isomorphism*

$$\mathrm{Sel}_{p^\infty}(E/K_\infty)^\vee \sim \big((\Lambda/J)^{\oplus e_1} \oplus \cdots \oplus (\Lambda/J^i)^{\oplus e_i} \oplus \cdots\big) \oplus \Lambda^{\oplus e_\infty} \oplus M'$$

with M' *a torsion* Λ*-module with characteristic ideal prime to* J.

Proof. This follows from Theorem 4.2 and Corollary 4.3 of [How04] when $N^- = 1$. We explain how to extend the result to squarefree N^- under the above hypothesis on $E[p]$.

Following the discussion in [*op.cit.*, §3] and adopting the notations there, we see that it suffices to show the vanishing of

$$\mathrm{H}^1_{\mathrm{ur}}(K_v, \mathbf{S}[p^k]) := \ker\big(\mathrm{H}^1(K_v, \mathbf{S}[p^k]) \to \mathrm{H}^1(K_v^{\mathrm{ur}}, \mathbf{S}[p^k])\big). \tag{4.1}$$

for every prime $v \nmid p$ inert in K, where $\mathbf{S}[p^k] = \varinjlim_n \mathrm{Ind}_{K_n/K} E[p^k]$. Since such primes v split completely in K_∞/K, by Shapiro's lemma and inflation-restriction we find

$$\begin{aligned} \mathrm{H}^1_{\mathrm{ur}}(K_v, \mathbf{S}[p^k]) &\simeq \ker\big(\mathrm{H}^1(K_v, E[p^k]) \otimes \Lambda^\vee \to \mathrm{H}^1(K_v^{\mathrm{ur}}, E[p^k]) \otimes \Lambda^\vee\big) \\ &\simeq \mathrm{H}^1(\mathbf{F}_v, E[p^k]^{I_v}) \otimes \Lambda^\vee \\ &= (E[p^k]^{I_v}/(\mathrm{Fr}_v - 1)E[p^k]^{I_v}) \otimes \Lambda^\vee, \end{aligned} \tag{4.2}$$

where $\mathbf{F}_v$ is the residue field of K_v, Fr_v is a Frobenius element at v, and $\Lambda^\vee = \mathrm{Hom}_{\mathbf{Z}_p}(\Lambda, \mathbf{Q}_p/\mathbf{Z}_p)$.

Since N^- is squarefree, any prime v as above is a prime of multiplicative reduction for E, so by Tate's uniformization we have

$$E[p^\infty] \sim \begin{pmatrix} \varepsilon & * \\ 0 & 1 \end{pmatrix}$$

as G_{K_v}-modules, where ε is the p-adic cyclotomic character. Since $\bar\rho_{E,p}$ is ramified at v, the image of '$*$' in the above matrix generates $\mathbf{Q}_p/\mathbf{Z}_p$. Thus we see that

$$E[p^\infty]^{I_v}/(\mathrm{Fr}_v - 1)E[p^\infty]^{I_v} = 0,$$

which by (4.2) implies the vanishing of $\mathrm{H}^1_{\mathrm{ur}}(K_v, \mathbf{S}[p^k])$. □

We conclude this section by recalling Howard's abstract generalization of Rubin's height formula for derived p-adic heights. For every prime v of K above p, let $\mathscr{F}_v^+ T_pE$ be the kernel of the reduction map $T_pE \to T_p\tilde{E}$, where $\tilde{E}$ is the reduction of E modulo v. Letting $V = V_pE$, this induces the filtration $\mathscr{F}_v^+ V \subset V$. For every prime $v|p$ of K write

$$\widehat{\mathrm{H}}^1_{\mathrm{fin}}(K_{\infty,v}, V) = \bigoplus_{w|v} \widehat{\mathrm{H}}^1(K_{\infty,w}, \mathscr{F}_v^+ V),$$

where w runs over the places of K_∞ above v. The local pairings in (3.6) induce a semi-local pairing

$$\langle -, - \rangle_{K_{\infty,v}} : \widehat{\mathrm{H}}^1(K_{\infty,v}, V) \times \widehat{\mathrm{H}}^1_{\mathrm{fin}}(K_{\infty,v}, V) \to \Lambda \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

which induces a perfect duality between the $\widehat{\mathrm{H}}^1(K_{\infty,v}, V)/\widehat{\mathrm{H}}^1_{\mathrm{fin}}(K_{\infty,v}, V)$ and $\widehat{\mathrm{H}}^1_{\mathrm{fin}}(K_{\infty,v}, V)$. Every class $\mathbf{z} \in \widehat{\mathrm{H}}^1(K_\infty, V)$ defines a linear map

$$\mathcal{L}_{p,\mathbf{z}} = \sum_{v|p} \langle \mathrm{Loc}_v(\mathbf{z}), - \rangle_{K_{\infty,v}} : \widehat{\mathrm{H}}^1_{\mathrm{fin}}(K_{\infty,p}, V) = \bigoplus_{v|p} \widehat{\mathrm{H}}^1_{\mathrm{fin}}(K_{\infty,v}, V) \to \Lambda \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,$$

Let $\mathrm{ord}(\mathcal{L}_{p,\mathbf{z}})$ be the largest integer r such that the image of $\mathcal{L}_{p,\mathbf{z}}$ is contained in J^r.

Theorem 4.2. *Suppose* $0 < r \leqslant \mathrm{ord}(\mathcal{L}_{p,\mathbf{z}})$. *Then* $z = \mathrm{pr}_K(\mathbf{z})$ *belongs to* $S_p^{(r)}(E/K)$ *and for any* $w \in S_p^{(r)}(E/K)$*, we have*

$$h_p^{(r)}(z, w) = -\mathcal{L}_{p,\mathbf{z}}(\mathbf{w}_p) \pmod{J^{r+1}}$$

where $\mathbf{w}_p = (\mathbf{w}_v)_{v|p} \in \widehat{\mathrm{H}}^1_{\mathrm{fin}}(K_{\infty,p}, V)$ *is any semi-local class with* $\mathrm{pr}_{K_v}(\mathbf{w}_v) = \mathrm{Loc}_v(w)$*,* $v|p$.

Proof. This is a reformulation of part (c) of Theorem 2.5 in [How04]. Note that the existence of $\mathbf{w}_p$ follows from the definition of $S_p^{(r)}(E/K)$ in *op.cit.*, and the fact that the image $\mathcal{L}_{p,\mathbf{z}}(\mathbf{w}_p) \in J^r/J^{r+1}$ is independent of the choice of $\mathbf{w}_p$ is shown in the proof. □

4.2. **Derived p-adic heights and the Coleman map.** Now we compute the local expression in Theorem 4.2 for the derived p-adic height pairing in terms of the Coleman map from §3, yielding our higher rank generalization of Rubin's formula (Theorem 4.5), which in addition to playing a key role in the proof of our results, may be of independent interest.

We use the setting and notations introduced after Lemma 3.5. In particular, $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K, with $\mathfrak{p}$ the prime of K above p induced by our fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Let $\hat{K}_\infty$ be the closure of the image of K_∞ in $\overline{\mathbf{Q}}_p$ under this embedding, and put

$$\Gamma_\infty = \mathrm{Gal}(K_\infty/K), \quad \hat{\Gamma}_\infty = \mathrm{Gal}(\hat{K}_\infty/\mathbf{Q}_p),$$

so naturally $\hat{\Gamma}_\infty$ is a subgroup of Γ_∞. Also, we put $F = \hat{H}_c$ for some fixed c prime to p, and $F_\infty = \hat{H}_{cp^\infty}$, which is a finite extension of $\hat{K}_\infty$.

Let $\boldsymbol{e} \in R^{\psi_{\mathcal{F}}=0}$ be a generator over $\mathscr{O}[\![G_\infty]\!]$ such that $\boldsymbol{e}(0) = 1$. Define

$$\mathbf{w}^\eta = \Omega^\epsilon_{V,1}(\eta \otimes \boldsymbol{e}) \in \widehat{\mathrm{H}}^1(F_\infty, V), \tag{4.3}$$

where $\Omega^\epsilon_{V,1}$ in is the big exponential map in Theorem 3.2.

As in §3.3, we let $\sigma_0 \in \mathrm{Gal}(F_\infty^{\mathrm{ur}}/\mathbf{Q}_p)$ be such that $\sigma_0|_{\mathbf{Q}_p^{\mathrm{ur}}} = \mathrm{Fr}$ is the absolute Frobenius.

Proposition 4.3. *Let* $\mathbf{Q}_p^{\mathrm{cyc}}$ *be the cyclotomic* $\mathbf{Z}_p^\times$*-extension of* $\mathbf{Q}_p$. *Let* $\sigma_{\mathrm{cyc}} \in \mathrm{Gal}(F_\infty^{\mathrm{ur}}/\mathbf{Q}_p)$ *be the Frobenius such that* $\sigma_{\mathrm{cyc}}|_{\mathbf{Q}_p^{\mathrm{cyc}}} = 1$ *and* $\sigma_{\mathrm{cyc}}|_{\mathbf{Q}_p^{\mathrm{ur}}} = \mathrm{Fr}$. *For each* $\hat{\mathbf{z}} \in \widehat{\mathrm{H}}^1(\hat{K}_\infty, V)$*, we have*

$$\langle \hat{\mathbf{z}}, \mathrm{cor}_{F_\infty/\hat{K}_\infty}(\mathbf{w}^\eta) \rangle_{\hat{K}_\infty} = \mathrm{pr}_{\hat{K}_\infty}(\mathrm{Col}^\eta(\hat{\mathbf{z}})) \sum_{i=1}^{[F:\mathbf{Q}_p]} \frac{\sigma^i_{\mathrm{cyc}}|_{\hat{K}_\infty}}{[F_\infty : \hat{K}_\infty] \cdot h_{\boldsymbol{e}}^{\mathrm{Fr}^i}} \in \mathcal{W}[\![\hat{\Gamma}_\infty]\!] \otimes \mathbf{Q}_p.$$

Proof. We first recall that for every $e \in (R \otimes_{\mathscr{O}} \mathcal{W})^{\psi_{\mathcal{F}}=0}$, the big exponential map $\Omega^{\epsilon}_{V,1}(\eta \otimes e)$ in Theorem 3.2 is given by

$$\Omega^{\epsilon}_{V,1}(\eta \otimes e) = (\exp_{F_n,V}(\Xi_{n,V}(G_e)))_{n=0,1,2,\ldots}, \tag{4.4}$$

where $G_e \in \mathbf{D}(V) \otimes \mathscr{H}_{1,\mathcal{Q}}(X)$ is a solution of $(1 - \varphi \otimes \varphi_{\mathcal{F}})G_e = \eta \otimes e$ and $\Xi_{n,V}$ is as in (3.4). Taking

$$G_e = G_{\boldsymbol{e}} = \sum_{m=0}^{\infty} (\varphi \otimes \varphi_{\mathcal{F}})^m (\eta \otimes \boldsymbol{e}) = \sum_{m=0}^{\infty} \varphi^m \eta \otimes \boldsymbol{e}^{\mathrm{Fr}^m},$$

we obtain

$$\begin{aligned} \Xi_{n,V}(G_{\boldsymbol{e}}) &= p^{-(n+1)}(\varphi^{-(n+1)} \otimes 1) G_{\boldsymbol{e}}^{\mathrm{Fr}^{-(n+1)}}(\epsilon_n) \\ &= \sum_{m=0}^{\infty} (p\varphi)^{-(n+1)} \varphi^m \eta \otimes \boldsymbol{e}^{\mathrm{Fr}^{m-(n+1)}}(\epsilon_{n-m}). \end{aligned} \tag{4.5}$$

Put $z_n = \mathrm{pr}_{\hat{K}_n}(\hat{\mathbf{z}})$ and $\hat{G}_n = \mathrm{Gal}(\hat{K}_n/\mathbf{Q}_p)$. From the definition of the Coleman map $\mathrm{Col}^{\epsilon}_{e}$, and using in (4.4) and (4.5), we thus find that

$$\begin{aligned} &\left[\mathrm{pr}_{\hat{K}_n}(\mathrm{Col}^{\epsilon}_{\boldsymbol{e}}(\hat{\mathbf{z}})), \eta\right]_V = \\ &\sum_{m=0}^{\infty} \left[\sum_{\gamma \in \hat{G}_n} \exp^*_{\hat{K}_n,V}(z_n^{\gamma^{-1}\sigma_0^{n+1-m}})\gamma, \sum_{\tau \in \hat{G}_n} (p\varphi)^{-(n+1)} \varphi^m \eta \otimes \boldsymbol{e}^{\mathrm{Fr}^{m-(n+1)}}(\epsilon_{n-m})^{\tau\sigma_0^{n+1-m}} \tau|_{\hat{K}_n} \right]_V, \end{aligned} \tag{4.6}$$

where $\exp^*_{\hat{K}_n,V}$ is the Bloch–Kato dual exponential map.

On the other hand, it is immediately seen that

$$\mathrm{pr}_{\hat{K}_n}(\langle \hat{\mathbf{z}}, \mathrm{cor}_{F_\infty/\hat{K}_\infty}(\mathbf{w}^\eta)\rangle_{\hat{K}_\infty}) = \frac{1}{[F_\infty : \hat{K}_\infty]} \sum_{j=1}^{[F:\mathbf{Q}_p]} \mathrm{pr}_{\hat{K}_n}(\langle \hat{\mathbf{z}}^{\sigma_0^{-j}}, \mathbf{w}^\eta \rangle_{F_\infty}) \sigma_0^j|_{\hat{K}_n},$$

and from (4.6) we find that

$$\begin{aligned} &\mathrm{pr}_{\hat{K}_n}(\langle \hat{\mathbf{z}}^{\sigma_0^{-j}}, \mathbf{w}^\eta \rangle_{F_\infty}) = \sum_{\gamma \in \hat{G}_n} \langle z_n^{\sigma_0^{-j}\gamma^{-1}}, \exp_{F_n,V}(\Xi_{n,V}(G_{\boldsymbol{e}})\rangle_{F_n} \gamma|_{\hat{K}_n} \\ &= \mathrm{Tr}_{F_n/\mathbf{Q}_p}\left(\left[\sum_{\gamma \in \hat{G}_n} \exp^*_{\hat{K}_n,V}(z_n^{\sigma_0^{-j}\gamma^{-1}})\gamma|_{\hat{K}_\infty}, \Xi_{n,V}(G_{\boldsymbol{e}}) \right]_V \right) \\ &= \sum_{m=0}^{\infty} \sum_{i=1}^{[F:\mathbf{Q}_p]} \left[\sum_{\gamma \in \hat{G}_n} \exp^*_{\hat{K}_n,V}(z_n^{\gamma^{-1}\sigma_0^{i-j+n+1-m}})\gamma, \sum_{\tau \in \hat{G}_n} (p\varphi)^{-(n+1)} \varphi^m \eta \otimes \boldsymbol{e}^{\mathrm{Fr}^{m-(n+1)}}(\epsilon_{n-m})^{\tau\sigma_0^{i+n+1-m}} \tau|_{\hat{K}_n} \right] \\ &= \sum_{i=1}^{[F:\mathbf{Q}_p]} \left[\mathrm{pr}_{\hat{K}_n}(\mathrm{Col}^{\epsilon}_{\boldsymbol{e}}(\mathbf{z}^{\sigma_0^{-j}})^{\sigma_0^i}), \eta\right]. \end{aligned}$$

Taking the limit over n, we thus arrive at

$$\begin{aligned} \langle \hat{\mathbf{z}}, \mathrm{cor}_{F_\infty/\hat{K}_\infty}(\mathbf{w}^\eta)\rangle_{\hat{K}_\infty} &= \frac{1}{[F_\infty : \hat{K}_\infty]} \sum_{j=1}^{[F:\mathbf{Q}_p]} \sum_{i=1}^{[F:\mathbf{Q}_p]} \left[\mathrm{pr}_{\hat{K}_\infty}(\mathrm{Col}^{\epsilon}_{\boldsymbol{e}}(\hat{\mathbf{z}}^{\sigma_0^{-j}})^{\sigma_0^i}), \eta\right] \sigma_0^j \\ &= \frac{1}{[F_\infty : \hat{K}_\infty]} \sum_{i=1}^{[F:\mathbf{Q}_p]} \mathrm{pr}_{\hat{K}_\infty}(\mathrm{Col}^{\eta}(\hat{\mathbf{z}})^{\sigma_0^i}) \cdot \frac{1}{h_{\boldsymbol{e}}^{\sigma_0^i}}, \end{aligned} \tag{4.7}$$

using (3.9) for the second equality. Finally, writing $g_\rho = \varphi(1+X)$ for the isomorphism φ in (3.8), one has $g_\rho^{\sigma_0^{-i}}(\epsilon_{i-1}) = \zeta_{p^i} \in \mathbf{Q}_p^{\mathrm{cyc}}$, which immediately implies the relation

$$\mathrm{pr}_{\hat{K}_\infty}(\mathrm{Col}^\eta(\hat{\mathbf{z}})) \cdot \sigma_{\mathrm{cyc}}^i = \mathrm{pr}_{\hat{K}_\infty}(\mathrm{Col}^\eta(\hat{\mathbf{z}})^{\sigma_0^i}).$$

Together with (4.7), this concludes the proof. □

We shall also need the following result.

Lemma 4.4. *The projection of* $\mathbf{w}^\eta$ *to* $\mathrm{H}^1(F,V)$ *is given by*

$$\mathrm{pr}_F(\mathbf{w}^\eta) = \exp_{F,V}\left(\frac{1-p^{-1}\varphi^{-1}}{1-\varphi}\eta\right).$$

Proof. Let $g = \eta \otimes \boldsymbol{e}$ and let $G(X) \in \mathbf{D}(V) \otimes \mathscr{H}_{1,\mathcal{Q}}(X)$ such that $(1-\varphi\otimes\varphi_{\mathcal{F}})G = g$. Then

$$G(\epsilon_0) = \eta \otimes \boldsymbol{e}(\epsilon_0) - \eta + (1-\varphi)^{-1}\eta,$$

and by definition,

$$\mathrm{pr}_F(\mathbf{w}^\eta) = \mathrm{cor}_{F_0/F}(\Xi_{0,V}(G)), \tag{4.8}$$

where $\Xi_{0,V}(G)$ is as in (3.4). Equation (3.1) and the fact that $\psi_{\mathcal{F}}\boldsymbol{e}(X) = 0$ imply that

$$\sum_{\zeta \in \mathcal{F}^{\mathrm{Fr}^{-1}}[p]} \boldsymbol{e}^{\mathrm{Fr}^{-1}}(X \oplus_{\mathcal{F}} \zeta) = 0,$$

from where we obtain

$$\mathrm{Tr}_{F_0/F}(G^{\mathrm{Fr}^{-1}}(\epsilon_0)) = \sum_{\tau \in \mathrm{Gal}(F_0/F)} \eta \otimes \boldsymbol{e}(\epsilon_0^\tau) - \eta + (1-\varphi)^{-1}\eta = \frac{p\varphi - 1}{1-\varphi}\eta.$$

Together with (4.8), we thus see that

$$\mathrm{pr}_F(\mathbf{w}^\eta) = \exp_{F,V}\mathrm{Tr}_{F_0/F}\left(p^{-1}\varphi^{-1}(G^{\mathrm{Fr}^{-1}}(\epsilon_0))\right) = \exp_{F,V}\left((1-p^{-1}\varphi^{-1})(1-\varphi)^{-1}\eta\right),$$

concluding the proof. □

Recall the identification $K_\mathfrak{p} = \mathbf{Q}_p$, and let $\mathrm{H}^1_{\mathrm{fin}}(\mathbf{Q}_p, V) \subset \mathrm{H}^1(\mathbf{Q}_p, V)$ be the subspace given by $\mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_\mathfrak{p}^+ V)$. As is well-known, $\mathrm{H}^1_{\mathrm{fin}}(\mathbf{Q}_p, V)$ agrees with the Bloch–Kato finite subspace. Let $\log_{\mathbf{Q},V} : \mathrm{H}^1_{\mathrm{fin}}(\mathbf{Q}_p, V) \to \mathbf{D}(V)$ be the Bloch–Kato logarithm map, and denote by $\log_{\omega_E,\mathfrak{p}}$ the composition

$$\log_{\omega,\mathfrak{p}} : \mathrm{H}^1(\mathbf{Q}_p, V) \xrightarrow{\log_{\mathbf{Q},V}} \mathbf{D}(V) \xrightarrow{\langle -,\omega_E \otimes t^{-1}\rangle_{\mathrm{dR}}} \mathbf{Q}_p \tag{4.9}$$

For a global class $\mathbf{z} \in \widehat{\mathrm{H}}^1(K_\infty, V)$, put

$$\mathrm{Col}^\eta(\mathrm{Loc}_\mathfrak{p}(\mathbf{z})) := \sum_{\sigma \in \Gamma_\infty/\hat{\Gamma}_\infty} \mathrm{Col}^\eta(\mathrm{Loc}_\mathfrak{P}(\mathbf{z}^{\sigma^{-1}}))\sigma \in \mathcal{W}[\![\Gamma_\infty]\!], \tag{4.10}$$

where $\mathrm{Loc}_\mathfrak{P} : \widehat{\mathrm{H}}^1(K_\infty, V) \to \widehat{\mathrm{H}}^1(\hat{K}_\infty, V)$ is the restriction map, and let J be the augmentation ideal of $\mathcal{W}[\![\Gamma_\infty]\!]$.

Theorem 4.5. *Let* $\mathbf{z} \in \widehat{\mathrm{H}}^1(K_\infty, V)$*, and denote by* $\mathfrak{r}$ *be the largest integer* r *such that*

$$\mathrm{Col}^\eta(\mathrm{Loc}_\mathfrak{p}(\mathbf{z})) \in J^r \quad \textit{and} \quad \mathrm{Col}^\eta(\mathrm{Loc}_\mathfrak{p}(\overline{\mathbf{z}})) \in J^r,$$

where $\overline{\mathbf{z}} = \mathbf{z}^\tau$ *for the complex conjugation* $\tau \in \mathrm{Gal}(K/\mathbf{Q})$*. Then for every* $0 < r \leqslant \mathfrak{r}$*, the class* $z = \mathrm{pr}_K(\mathbf{z})$ *belongs to* $S_p^{(r)}(E/K)$ *and for every* $x \in S_p^{(r)}(E/K)$*, we have*

$$h_p^{(r)}(z,x) = -\frac{1-p^{-1}\alpha_p}{1-\alpha_p^{-1}} \cdot \left(\mathrm{Col}^\eta(\mathrm{Loc}_\mathfrak{p}(\mathbf{z})) \cdot \log_{\omega,\mathfrak{p}}(x) + \mathrm{Col}^\eta(\mathrm{Loc}_\mathfrak{p}(\overline{\mathbf{z}})) \cdot \log_{\omega,\mathfrak{p}}(\overline{x})\right) \pmod{J^{r+1}},$$

where $\overline{x} = x^\tau$.

Proof. The inclusion $z \in S_p^{(r)}(E/K)$ follows immediately from Theorem 4.2. Let $x \in S_p^{(r)}(E/K)$, and put

$$\mathbf{w}_{\mathfrak{P}} := \mathrm{cor}_{F_\infty/\hat{K}_\infty}(\mathbf{w}^\eta) \in \widehat{\mathrm{H}}^1_{\mathrm{fin}}(\hat{K}_\infty, V).$$

Then, since $\dim_{\mathbf{Q}_p} \mathrm{H}^1_{\mathrm{fin}}(\mathbf{Q}_p, V) = 1$, we can write

$$\mathrm{Loc}_{\mathfrak{p}}(x) = c \cdot \mathrm{pr}_{\mathbf{Q}_p}(\mathbf{w}_{\mathfrak{P}})$$

for some $c \in \mathbf{Q}_p$. Since $\mathrm{pr}_{\mathbf{Q}_p}(\mathbf{w}_{\mathfrak{P}}) = \mathrm{cor}_{F/\mathbf{Q}_p}(\mathbf{w}^\eta)$, from Lemma 4.4 and (3.14) we see that

$$\langle \log_{\mathbf{Q}_p,V}(\mathrm{pr}_{\mathbf{Q}_p}(\mathbf{w}_{\mathfrak{P}})), \omega_E \otimes t^{-1}\rangle_{\mathrm{dR}} = [F:\mathbf{Q}_p] \cdot \frac{1-\alpha_p^{-1}}{1-p^{-1}\alpha_p},$$

from where we deduce that

$$c = \frac{1-p^{-1}\alpha_p}{1-\alpha_p^{-1}} \cdot [F:\mathbf{Q}_p]^{-1} \cdot \log_{\omega_E,\mathfrak{p}}(x).$$

Together with the formula in Theorem 4.2, this gives the equality

$$h_p^{(r)}(z,x) = -\frac{1-p^{-1}\alpha_p}{1-\alpha_p^{-1}} \cdot [F:\mathbf{Q}_p]^{-1}$$
$$\times \left(\sum_{\sigma \in \Gamma_\infty/\hat{\Gamma}_\infty} \log_{\omega_E,\mathfrak{p}}(x) \cdot \langle \mathrm{Loc}_{\mathfrak{P}}(\mathbf{z}^{\sigma^{-1}}), \mathbf{w}_{\mathfrak{P}}\rangle_{\hat{K}_\infty} \sigma + \log_{\omega_E,\mathfrak{p}}(\overline{x}) \cdot \langle \mathrm{Loc}_{\mathfrak{P}}(\overline{\mathbf{z}}^{\sigma^{-1}}), \mathbf{w}_{\mathfrak{P}}\rangle_{\hat{K}_\infty} \sigma \right)$$

in J^r/J^{r+1}. Since $h_{\boldsymbol{e}} \equiv 1 \pmod{J}$, as is immediate from the defining relation $\rho(1+X) = h_{\boldsymbol{e}} \cdot \boldsymbol{e}$ and the fact that $\boldsymbol{e}(0) = 1$, the result now follows from Proposition 4.3. □

5. Proof of Theorem A

We begin by recalling the setting before concluding the proof. Let $E/\mathbf{Q}$ be an elliptic curve of conductor N with good ordinary reduction at $p > 3$, and assume that E has root number $+1$ and $L(E,1) = 0$ (so, of course, $\mathrm{ord}_{s=1} L(E,s) \geq 2$). Let K be an imaginary quadratic field of discriminant prime to N in which $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ splits, with $\mathfrak{p}$ the prime of K above p induced by our fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Let ψ be a ray class character of K of conductor prime to Np, and as in Conjecture 1.2 assume that

(a) $L(E^K,1) \cdot L(E/K,\chi,1) \neq 0$,
(b) $\chi(\overline{\mathfrak{p}}) \neq 1$,

where $\chi = \psi/\psi^\tau$. In addition, we assume that

(c) $E[p]$ is irreducible as a $G_{\mathbf{Q}}$-module,
(d) N^- is square-free,
(e) $E[p]$ is ramified at every prime $q | N^-$,

where N^- is the maximal factor of N divisible only by primes inert in K. Let (f,g,g^*) be the triple consisting of the newform $f \in S_2(\Gamma_0(N))$ associated to E and the weight one theta series associated to ψ and ψ^{-1}, respectively. Finally, put $\alpha = \psi(\overline{\mathfrak{p}})$ and $\beta = \psi(\mathfrak{p})$.

5.1. Generalized Kato classes.

By construction, the Hida families

$$\boldsymbol{g} = \boldsymbol{g}_\alpha = \boldsymbol{\theta}_\psi(S), \quad \boldsymbol{g}^* = \boldsymbol{g}^*_{\alpha^{-1}} = \boldsymbol{\theta}_{\psi^{-1}}(S) \in \mathcal{O}[\![S]\!][\![q]\!]$$

considered in §2.4 specialize at $S = \mathbf{v}-1$ to g_α and $g^*_{\alpha^{-1}}$, the p-stabilizations of g and g^* with U_p-eigenvalue α and α^{-1}, respectively. Thus for every choice of test vectors $(\breve{f}, \breve{\boldsymbol{g}}_\alpha, \breve{\boldsymbol{g}}^*_{\alpha^{-1}})$ the $\mathcal{O}[\![S]\!]$-adic class $\kappa(\breve{f}, \breve{\boldsymbol{g}}_\alpha \breve{\boldsymbol{g}}^*_{\alpha^{-1}})$ in (3.11) specializes to the *generalized Kato class*

$$\kappa_{\alpha,\alpha^{-1}}(f,g,g^*) := \kappa(\breve{f}, \breve{\boldsymbol{g}}_\alpha \breve{\boldsymbol{g}}^*_{\alpha^{-1}})|_{S=\mathbf{v}-1} \in \mathrm{H}^1(\mathbf{Q}, V_{fgg^*}),$$

where $V_{fgh} := V_f \otimes V_g \otimes V_h$.

Varying over the possible combinations of roots of the Hecke polynomial at p for g and g^*, we thus obtain the four generalized Kato classes

$$\kappa_{\alpha,\alpha^{-1}}(f,g,g^*),\ \kappa_{\alpha,\beta^{-1}}(f,g,g^*),\ \kappa_{\beta,\alpha^{-1}}(f,g,g^*),\ \kappa_{\beta,\beta^{-1}}(f,g,g^*)\in \mathrm{H}^1(\mathbf{Q},V_{fgg^*}). \tag{5.1}$$

Note the $G_{\mathbf{Q}}$-module decomposition (1.7) yields

$$\begin{aligned}\mathrm{H}^1(\mathbf{Q},V_{fgg^*}) &\simeq \mathrm{H}^1(\mathbf{Q},V_pE)\oplus \mathrm{H}^1(\mathbf{Q},V_pE\otimes \mathrm{ad}^0V_p(g))\\ &\simeq \mathrm{H}^1(\mathbf{Q},V_pE)\oplus \mathrm{H}^1(\mathbf{Q},V_pE^K)\oplus \mathrm{H}^1(K,V_pE\otimes\chi),\end{aligned}$$

where E^K is the twist of E by the quadratic character corresponding to K.

Lemma 5.1. *The projections to* $\mathrm{H}^1(\mathbf{Q},V_pE)$ *of each of the classes in* (5.1) *lands in* $\mathrm{Sel}(\mathbf{Q},V_pE)$.

Proof. Since we are assuming $L(E,1)=0$ and (a) above, the result follows from the vanishing of $\mathrm{Sel}(\mathbf{Q},V_pE^K)$ and $\mathrm{Sel}(K,V_pE\otimes\chi)$ by the same argument as in Lemma 3.5. □

5.2. **Vanishing of $\kappa_{\alpha,\beta^{-1}}(f,g,g^*)$ and $\kappa_{\beta,\alpha^{-1}}(f,g,g^*)$.** This part follows easily from the work of Darmon–Rotger [DR21] and Bertolini–Seveso–Venerucci [BSV21].

Proposition 5.2. $\kappa_{\alpha,\beta^{-1}}(f,g,g^*)=\kappa_{\beta,\alpha^{-1}}(f,g,g^*)=0.$

Proof. Let

$$\boldsymbol{g}_\alpha=\boldsymbol{\theta}_{\psi,\alpha}(S_2)\in\mathcal{O}[\![S_2]\!][\![q]\!],\quad \boldsymbol{g}^*_{\beta^{-1}}=\boldsymbol{\theta}_{\psi^{-1},\beta^{-1}}(S_3)\in\mathcal{O}[\![S_3]\!][\![q]\!]$$

be CM Hida families as in §2.4, but passing through the specialization $(g_\alpha,g_{\beta^{-1}})$ rather than $(g_\alpha,g_{\alpha^{-1}})$. Let

$$\kappa(f,\boldsymbol{g}_\alpha\boldsymbol{g}^*_{\beta^{-1}})(S_2,S_3)\in\mathrm{H}^1(\mathbf{Q},\mathbb{V}^\dagger_{f\boldsymbol{g}_\alpha\boldsymbol{g}^*_{\beta^{-1}}}) \tag{5.2}$$

be the two-variable restriction of the three-variable cohomology class constructed in [DR21] and [BSV21] (after a choice of level-N test vectors $\breve{\boldsymbol{g}}_\alpha$, $\breve{\boldsymbol{g}}^*_{\beta^{-1}}$ that we omit from the notation), and consider the further restriction

$$\boldsymbol{\kappa}^\iota:=\kappa(f,\boldsymbol{g}_\alpha\boldsymbol{g}^*_{\beta^{-1}})(\mathbf{v}(1+T)-1,\mathbf{v}(1+T)^{-1}-1)\in\mathrm{H}^1(\mathbf{Q},\mathbb{V}^\dagger_{f\boldsymbol{g}_\alpha(\boldsymbol{g}^*_{\beta^{-1}})^\iota}),$$

where $\mathbb{V}^\dagger_{f\boldsymbol{g}_\alpha(\boldsymbol{g}^*_{\beta^{-1}})^\iota}\simeq(V_pE\otimes\mathrm{Ind}_K^{\mathbf{Q}}\chi)\oplus(V_pE\otimes\mathrm{Ind}_K^{\mathbf{Q}}\Psi_T^{1-\tau})$. Thus $\boldsymbol{\kappa}^\iota$ is the restriction of (5.2) to the line of weights $(\ell,2-\ell)$ (*cf.* $\kappa(\breve{f},\breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*)$ in (3.11), where the line (ℓ,ℓ) is considered).

By definition, we have the equality

$$\kappa_{\alpha,\beta^{-1}}(f,g,g^*)=\boldsymbol{\kappa}^\iota(\mathbf{v}-1,\mathbf{v}-1).$$

As in Theorem 3.6, by [DR21, Prop. 5.8] the restriction $\mathrm{Loc}_p(\boldsymbol{\kappa}^\iota)$ belongs to the natural image of $\mathrm{H}^1(\mathbf{Q}_p,\mathscr{F}^{++}\mathbb{V}^\dagger_{f\boldsymbol{g}_\alpha(\boldsymbol{g}^*_{\beta^{-1}})^\iota})$ in $\mathrm{H}^1(\mathbf{Q}_p,\mathbb{V}^\dagger_{f\boldsymbol{g}_\alpha(\boldsymbol{g}^*_{\beta^{-1}})^\iota})$, where

$$\mathscr{F}^{++}\mathbb{V}^\dagger_{f\boldsymbol{g}_\alpha(\boldsymbol{g}^*_{\beta^{-1}})^\iota}=V_pE\otimes\chi^{-1}+\mathscr{F}^+V_pE\otimes(\Psi_T^{1-\tau}+\Psi_T^{1-\tau}).$$

Thus the projection $\boldsymbol{\kappa}^\iota_\infty$ of $\boldsymbol{\kappa}^\iota$ to $\mathrm{H}^1(\mathbf{Q},V_pE\otimes\mathrm{Ind}_K^{\mathbf{Q}}\Psi_T^{1-\tau})\simeq\widehat{\mathrm{H}}^1(K_\infty,V_pE)$ is crystalline at p, and therefore defines a Selmer class for V_pE over the K_∞/K. Since under our hypotheses the space of such anticyclotomic universal norms is trivial by Cornut–Vatsal [CV05], we conclude that $\boldsymbol{\kappa}^\iota_\infty=0$. As in the proof of Theorem 3.6, it follows that $\kappa_{\alpha,\beta^{-1}}(f,g,g^*)=0$. The vanishing of $\kappa_{\beta,\alpha^{-1}}(f,g,g^*)$ is shown in the same manner. □

5.3. **The leading term formula.** Let $J \subset \Lambda$ be the augmentation ideal, and let

$$\mathfrak{r} = \mathrm{ord}_J(\Theta_{f/K}) := \sup\{s \geq 0 \mid \Theta_{f/K} \in J^s\}.$$

Since $\Theta_{f/K}$ is nonzero by [Vat03], $\mathfrak{r}$ is a well-defined non-negative integer, and since $L(E/K,1) = 0$ under our hypotheses, $\rho > 0$ by the interpolation property. Let

$$\mathrm{Sel}(K, V_pE) = S_p^{(1)} \supset S_p^{(2)} \supset \cdots \supset S_p^{(i)} \supset S_p^{(i+1)} \supset \cdots \supset S_p^{(\infty)} = 0 \tag{5.3}$$

be the filtration in Theorem 4.1, where we have put $S_p^{(i)} = S_p^{(i)}(E/K)$ for the ease of notation, and let

$$h_p^{(i)} : S_p^{(i)} \times S_p^{(i)} \to (J^i/J^{i+1}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

be the associated derived p-adic height pairings. Note that the vanishing of $S_p^{(\infty)}$ follows from [CV05]. From Corollary 3.7 and Theorem 4.5 we obtain the following key result.

Theorem 5.3. *Let* $\mathfrak{r} = \mathrm{ord}_J(\Theta_{f/K})$. *Then*

$$\kappa_{\alpha,\alpha^{-1}}(f, g, g^*) \in S_p^{(\mathfrak{r})}, \tag{5.4}$$

and that for every for every $x \in S_p^{(\mathfrak{r})}$ *we have*

$$h_p^{(\mathfrak{r})}(\kappa_{\alpha,\alpha^{-1}}(f, g, g^*), x) = \frac{1 - p^{-1}\alpha_p}{1 - \alpha_p^{-1}} \cdot \Theta_{f/K} \cdot \log_{\omega_E,\mathfrak{p}}(x) \cdot C \pmod{J^{\mathfrak{r}+1}}, \tag{5.5}$$

where α_p *is the* p*-adic unit root of* $X^2 - a_p(E)X + p = 0$ *and* C *is a non-zero algebraic number with* $C^2 \in K(\chi, \alpha_p)^\times$.

5.4. **Non-vanishing of** $\kappa_{\alpha,\alpha^{-1}}(f, g, g^*)$**.** Here we prove the implication (1.10) in Theorem A. Thus suppose that $\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2$. Since $L(E^K, 1) \neq 0$, we have $\mathrm{Sel}(\mathbf{Q}, V_pE^K) = 0$ by [Kol88] (or, alternatively, [Kat04]), and therefore

$$\mathrm{Sel}(K, V_pE) = \mathrm{Sel}(\mathbf{Q}, V_pE), \quad (r^+, r^-) = (2, 0), \tag{5.6}$$

where $r^\pm$ denotes the dimension of the $\pm$-eigenspace of $\mathrm{Sel}(K, V_pE)$ under the action of complex conjugation τ. Since τ acts as -1 on J/J^2, part (4) of Theorem 4.1 gives

$$h_p^{(i)}(x^\tau, y^\tau) = (-1)^r h_p^{(i)}(x, y), \tag{5.7}$$

and hence from (5.6) we see that for i odd, the null-space of $h_p^{(i)}$ (i.e., $S_p^{(i+1)}$) is either zero or two-dimensional, with the latter case occurring as long as $S_p^{(i)} \neq 0$. Since on the other hand $h_p^{(i)}$ is a non-degenerate alternating pairing on $S_p^{(i)}/S_p^{(i+1)}$ for even values of i, unless $S_p^{(i)} = 0$, it follows that (5.3) reduces to

$$\mathrm{Sel}(\mathbf{Q}, V_pE) = S_p^{(1)} = S_p^{(2)} = \cdots = S_p^{(r)} \supsetneq S_p^{(r+1)} = \cdots = S_p^{(\infty)} = 0 \tag{5.8}$$

for some even $r \geqslant 2$. By Theorem 4.1, we deduce that there is a Λ-module pseudo-isomorphism

$$\mathrm{Sel}_{p^\infty}(E/K_\infty)^\vee \sim (\Lambda/J^r)^{\oplus 2} \oplus M',$$

where M' is a torsion Λ-module with characteristic ideal prime to J. Therefore letting $\mathcal{L}_p \in \Lambda$ be any generator of the characteristic ideal of $\mathrm{Sel}_{p^\infty}(E/K_\infty)^\vee$, we have

$$\mathrm{ord}_J(\mathcal{L}_p) = 2r.$$

Finally, the divisibility $(\Theta_{f/K}^2) \supset (\mathcal{L}_p)$ arising from [SU14, §3.6.3] implies that $r \geqslant \mathfrak{r}$, and hence $S_p^{(\mathfrak{r})} = \mathrm{Sel}(\mathbf{Q}, V_pE)$ by (5.8). Since by our hypothesis that $\mathrm{Sel}(\mathbf{Q}, V_pE) \neq \ker(\mathrm{Loc}_p)$ we can find $x \in \mathrm{Sel}(\mathbf{Q}, V_pE)$ with $\log_{\omega_E,\mathfrak{p}}(x) \neq 0$, the non-vanishing of $\kappa_{\alpha,\alpha^{-1}}(f, g, g^*)$ now follows from the leading term formula (5.5).

Remark 5.4. The same argument as above with β in place of α establishes the non-vanishing of $\kappa_{\beta,\beta^{-1}}(f, g, g^*)$ under the given hypotheses.

5.5. **Analogue of Kolyvagin's theorem for $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$.** Here we prove the implication (1.9) in Theorem A. As in §5.4, we see that $\mathrm{Sel}(K,V_pE)=\mathrm{Sel}(\mathbf{Q},V_pE)$ and the non-trivial jumps in (5.3) can only occur at even values of i. Thus (5.3) reduces to

$$\mathrm{Sel}(\mathbf{Q},V_pE)=S_p^{(1)}=\cdots=S_p^{(2r_1)}\supsetneq S_p^{(2r_1+1)}=\cdots=S_p^{(2r_t)}\supsetneq S_p^{(2r_t+1)}=\cdots=S_p^{(\infty)}=0 \tag{5.9}$$

for some $1\leqslant r_1\leqslant\cdots\leqslant r_t$, and by Theorem 4.1 we have

$$\mathrm{Sel}_{p^\infty}(E/K_\infty)^\vee\sim(\Lambda/J^{2r_1})^{d_1}\oplus\cdots\oplus(\Lambda/J^{2r_t})^{\oplus d_t}\oplus M'$$

where $d_i=\dim_{\mathbf{Q}_p}(S_p^{(2r_i)}/S_p^{(2r_i+1)})\geqslant 2$ and M' is as in §5.4. Letting $\mathcal{L}_p\in\Lambda$ be a generator of the characteristic ideal of $\mathrm{Sel}_{p^\infty}(E/K_\infty)^\vee$, we therefore have

$$\mathrm{ord}_J(\mathcal{L}_p)=2(r_1d_1+\cdots+r_td_t),\quad \dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q},V_pE)=d_1+\cdots+d_t. \tag{5.10}$$

Suppose now that $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)\neq 0$. By (5.4), it follows that $S_p^{(\mathfrak{r})}\neq 0$ and therefore

$$\mathfrak{r}\leqslant 2r_t. \tag{5.11}$$

On the other hand, the divisibility $(\mathcal{L}_p)\supset(\Theta_{f/K}^2)$ established in [BD05] (as refined in [PW11]) implies that $r_1d_1+\cdots+r_td_t\leqslant\mathfrak{r}$; together with (5.11) this yields

$$2r_t\geqslant r_1d_1+\cdots+r_td_t\geqslant 2(r_1+\cdots+r_t),$$

from which we conclude that $t=1$, $d_1=2$, and $\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q},V_pE)=2$.

5.6. **Application to the strong elliptic Stark conjecture.** We keep the setting from the beginning of this section, but assume in addition that $\#Ш(E/\mathbf{Q})[p^\infty]<\infty$.

As explained in [DR16, §4.5.3], the p-adic regulators appearing in the *elliptic Start conjectures* of [DLR15] all vanish in the setting we have placed ourselves in. As a remedy, in [DR16] they formulated a strengthening of those conjectures in terms of certain *enhanced regulators*; in our setting they are given (modulo $\mathbf{Q}^\times$) by

$$\mathrm{Log}_p(P\wedge Q)=P\otimes\log_p(Q)-Q\otimes\log_p(P)$$

where (P,Q) is any basis of $E(\mathbf{Q})\otimes_{\mathbf{Z}}\mathbf{Q}$. The strong elliptic Stark conjecture then predicts that the generalized Kato classes $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$ and $\kappa_{\beta,\beta^{-1}}(f,g,g^*)$ both agree with $\mathrm{Log}_p(P\wedge Q)$ up to a nonzero algebraic constant.

In the direction of this conjecture, our methods show that $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$ and $\kappa_{\beta,\beta^{-1}}(f,g,g^*)$ span the same p-adic line as $\mathrm{Log}_p(P\wedge Q)$ inside the 2-dimensional $\mathrm{Sel}(\mathbf{Q},V_pE)$.

To state the application, we identify $J^{\mathfrak{r}}/J^{\mathfrak{r}+1}$ with $\mathbf{Z}_p$ in the usual manner by choosing a topological generator of Γ_∞, and let $\Theta_{f/K}^{(\mathfrak{r})}\in\mathbf{Z}_p\smallsetminus\{0\}$ denote the image of $\Theta_{f/K}\pmod{J^{\mathfrak{r}+1}}$ under this identification.

Theorem 5.5. *Let the setting be as in the beginning of Section 5, and let $\mathfrak{r}=\mathrm{ord}_J(\Theta_{f/K})$. Then, as elements of $\mathrm{Sel}(\mathbf{Q},V_pE)\simeq E(\mathbf{Q})\otimes_{\mathbf{Z}}\mathbf{Q}_p$, we have*

$$\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)=C\cdot\frac{1-p^{-1}\alpha_p}{1-\alpha_p^{-1}}\cdot\frac{\Theta_{f/K}^{(\mathfrak{r})}}{h_p^{(\mathfrak{r})}(P,Q)}\cdot\mathrm{Log}_p(P\wedge Q),$$

where C is nonzero and such that $C^2\in K(\chi,\alpha_p)^\times$. The same result holds of $\kappa_{\beta,\beta^{-1}}(f,g,g^*)$.

Proof. Immediate from the leading term formula of §5.3 applied to $x=P$ and Q. □

Remark 5.6. The term $h_p^{(\mathfrak{r})}(P,Q)$ recovers the derived regulator R_{der} introduced in [BD95]. Thus Theorem 5.5 links the conjectural algebraicity of the ratio between $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$ and $\mathrm{Log}_p(P\wedge Q)$, as predicted in [DR16, §4.5.3], to a refinement of the p-adic Birch and Swinnerton-Dyer conjecture in [BD96, Conjecture 4.3] formulated in terms of R_{der}.

Appendix. Non-vanishing of $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$: Numerical examples

In this appendix, we applied Theorem A (particularly, the leading term formula in §5.3) to exhibit the first examples of non-vanishing generalized Kato classes for rational elliptic curves of rank 2.

Setting. In the examples tabulated below, we take elliptic curves $E/\mathbf{Q}$ with

$$\mathrm{ord}_{s=1}L(E,s) = 2 = \mathrm{rank}_{\mathbf{Z}}\, E(\mathbf{Q})$$

of conductor $N \in \{q, 2q\}$, with q an odd prime, and pairs $(p,-d)$ consisting of a prime $p > 3$ and a squarefree integer $-d < 0$ such that:

- $K = \mathbf{Q}(\sqrt{-d})$ has class number one, q is inert in K, and $L(E^K,1) \neq 0$,
- p splits in K and $E[p]$ is irreducible as a $G_{\mathbf{Q}}$-module.

Note that such pairs $(p,-d)$ can be easily produced. Indeed, [Rib90, Thm. 1.1] implies that $E[p]$ must ramify at $N^- = q$, and the irreducibility of $E[p]$ can be verified either by [Maz78] when $p \geqslant 11$ or by checking (from e.g. Cremona's tables) that E does not admit any rational m-isogenies for $m > 3$.

For every such triple $(E,p,-d)$, there is a ring class character χ of K of ℓ-power conductor for some prime $\ell \nmid Np$ such that $L(E/K,\chi,1) \neq 0$. (In fact, there are infinitely many such χ, as follows from [Vat03, Thm. 1.3] and its extension in [CH18, Thm. D].) Writing $\chi = \psi/\psi^\tau$ and letting $g = \theta_\psi$ and $g^* = \theta_{\psi^{-1}}$ we then have the class

$$\kappa_{\alpha,\alpha^{-1}}(f,g,g^*) \in \mathrm{Sel}(\mathbf{Q}, V_pE)$$

as in §5.2 (see Lemma 5.1).

Viewing $\Theta_{f/K}$ as an element in the power series ring $\mathbf{Z}_p[\![T]\!]$ as usual, in each of the examples below we checked that

$$\mathrm{ord}_T(\Theta_{f/K}) = 2. \tag{5.12}$$

By [BD05, Cor. 3] (using the extension of the main result of [BD05] contained in [PW11]), it follows that $\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2$. Since the condition $\mathrm{Sel}(\mathbf{Q}, V_pE) \neq \ker(\mathrm{Loc}_p)$ is automatic as long as $\#E(\mathbf{Q}) = \infty$, the non-vanishing of $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$ in these cases follows directly from the leading term formula of §5.3.

Verifying order of vanishing 2. Let us add some comments on the verification of (5.12) in the examples below. Let B be the definite quaternion algebra over $\mathbf{Q}$ of discriminant q, let $R \subset B$ be an Eichler order of level N/q, and let $\mathrm{Cl}(R)$ be the class group of R. Let

$$\phi_f : \mathrm{Cl}(R) \to \mathbf{Z}$$

be the Hecke eigenfunction associated to f by Jacquet–Langlands, normalized so that $\phi_f \not\equiv 0 \pmod{p}$. Fix an isomorphism $i_p : R \otimes \mathbf{Z}_p \simeq \mathrm{M}_2(\mathbf{Z}_p)$ and an optimal embedding $\mathcal{O}_K \hookrightarrow R$ such that K is sent to a subspace consisting of diagonal matrices, and for $a \in \mathbf{Z}_p^\times$ and $n \geqslant 0$ put

$$r_n(a) = i_p^{-1}(\begin{pmatrix} 1 & ap^{-n} \\ 0 & 1 \end{pmatrix}) \in \widehat{B}^\times,$$

where $\widehat{B} = B \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}$ is the adelic completion of B.

Consider the sequence $\{P_n^a\}_{n\geqslant 0}$ of right R-ideals given by $P_n^a := (r_n(a)\widehat{R}) \cap B$, and define the n-th theta element $\Theta_{f/K,n} \in \mathbf{Z}_p[T]$ by

$$\Theta_{f/K,n} := \frac{1}{\alpha_p^{n+1}} \sum_{i=0}^{p^n-1} \sum_{a\in\mu_{p-1}} \Big(\alpha_p \cdot \phi_f(P_n^{a\mathbf{u}^i}) - \phi_f(P_{n+1}^{a\mathbf{u}^i})\Big)\,(1+T)^i,$$

where α_p is the p-adic unit root of $x^2 - a_p(E)x + p$ and $\mathbf{u} = 1 + p$.

By the definition of $\Theta_{f/K}$ (see e.g. [BD96, §2.7]), we have

$$\Theta_{f/K} \equiv \Theta_{f/K,n} \pmod{(1+T)^{p^n}-1}.$$

Since $(p^n, (1+T)^{p^n}-1) \subset (p^n, T^p)$, in the examples listed in the following tables we could verify (5.12) by computing $\Theta_{f/K,n}$ mod (p^n, T^p) for $n = 2$ and 3, respectively. The computations were done using the Brandt module package in SAGE.

E	p	$-d$	$\Theta_{f/K}$ mod (p^2, T^p)
389a1	11	-2	$10T^2 + 69T^3 + T^4 + 103T^5 + 106T^6 + 66T^7 + 11T^8 + 55T^9 + 110T^{10}$
433a1	11	-7	$88T^2 + 22T^3 + 86T^4 + 7T^5 + 10T^6 + 12T^7 + 29T^8 + 88T^9 + 48T^{10}$
446c1	7	-3	$22T^2 + 27T^3 + 3T^4 + 16T^5 + 11T^6$
563a1	5	-1	$18T^2 + 9T^3 + 5T^4$
643a1	5	-1	$T^2 + 21T^4$
709a1	11	-2	$27T^2 + 114T^3 + 3T^4 + 14T^5 + 36T^6 + 15T^7 + 42T^8 + 44T^9 + 91T^{10}$
718b1	5	-19	$3T^2 + 20T^3 + 12T^4$
794a1	7	-3	$47T^2 + 23T^3 + 8T^4 + 24T^5 + 7T^6$
997b1	11	-2	$71T^2 + 41T^3 + 83T^4 + 19T^5 + 114T^6 + 111T^7 + 101T^8 + 46T^9 + 102T^{10}$
997c1	11	-2	$54T^2 + 38T^3 + 36T^4 + 81T^5 + 82T^6 + 18T^7 + 72T^8 + 95T^9 + 4T^{10}$
1034a1	5	-19	$22T^2 + 4T^3 + 6T^4$
1171a1	5	-1	$6T^2 + 6T^3 + 20T^4$
1483a1	13	-1	$128T^2 + 148T^3 + 127T^4 + 162T^5 + 30T^6 + 149T^7 + 141T^8 + 97T^9 + 49T^{10} + 13T^{11} + 29T^{12}$
1531a1	5	-1	$16T^2 + 7T^3 + 21T^4$
1613a1	17	-2	$128T^2 + 165T^3 + 224T^4 + 287T^5 + 140T^6 + 211T^7 + 147T^8 + 160T^9 + 59T^{10} + 122T^{11} + 195T^{12} + 43T^{13} + 207T^{14} + 214T^{15} + 285T^{16}$
1627a1	13	-1	$101T^2 + 151T^3 + 58T^4 + 104T^5 + 3T^6 + 165T^7 + 128T^8 + 63T^9 + 17T^{10} + 55T^{11} + 166T^{12}$
1907a1	13	-1	$72T^2 + 131T^3 + 32T^4 + 142T^5 + 84T^6 + 104T^7 + 90T^8 + 105T^9 + 38T^{10} + 92T^{11} + 116T^{12}$
1913a1	7	-3	$41T^2 + 16T^3 + 28T^4 + 23T^5 + 14T^6$
2027a1	13	-1	$54T^2 + 128T^3 + 65T^4 + 93T^5 + 83T^6 + 161T^7 + 113T^8 + 133T^9 + 49T^{10} + 151T^{11} + 13T^{12}$

E	p	$-d$	$\Theta_{f/K}$ mod (p^3, T^p)
571b1	5	-1	$100T^2 + 100T^3 + 15T^4$
1621a1	11	-2	$1089T^2 + 807T^4 + 986T^5 + 586T^6 + 1098T^7 + 772T^8 + 228T^9 + 1296T^{10}$

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