Michigan Math. J. 51 (2003)

Combinatorial Method in Adjoint Linear Systems on Toric Varieties

Hui-Wen Lin

1. Introduction

Inspired by minimal model theory, Fujita in the 1980s conjectured as follows.

Conjecture [6]. *Let X be a nonsingular complex projective variety of dimension n, and let D be an ample divisor on X. Then*:

(I) *$K_X + \ell D$ is generated by global sections for $\ell \geq n + 1$; and*
(II) *$K_X + \ell D$ is very ample for $\ell \geq n + 2$.*

Moreover, (I) *and* (II) *should still hold true if X has only "mild singularities".*

For nonsingular varieties, the one-dimensional case is an easy fact in curve theory. The two-dimensional case follows from the work of Reider [16]. In higher-dimensional cases, (I) is known for $n = 3$ [3] and $n = 4$ [8], and by [1] we know that $K_X + \frac{1}{2}(n^2 + n + 2)D$ is generated by global sections for all n. Less is known about (II) with one exception: if D is already very ample, then (I) and (II) follow from Bertini's theorem by induction on dimensions.

For part (I), allowing X to have rational Gorenstein singularities, Fujita himself had shown (among other things) that $K_X + (n + 1)D$ is nef. For varieties over a field of arbitrary characteristic that have singularities of F-rational type, Smith showed that (I) holds if D is further assumed to be generated by global sections [17]. (In characteristic zero this can also be proved by using vanishing theorems.) Both [6] and [17] apply well to quite general toric varieties, since they have only rational singularities and on them a Cartier divisor is nef if and only if it is basepoint free (cf. Section 5). Moreover, ample divisors are automatically generated by global sections (Corollary 2.3). In fact, for *nonsingular* toric varieties, Fujita's conjectures hold because ample divisors are automatically very ample (Demazure's theorem).

These implications motivate our present work: results on toric varieties should admit direct proofs using only toric (combinatorial) techniques. In this note such elementary proofs are found for rather general toric varieties. Moreover, our combinatorial treatment also provides results on the "very ampleness" conjecture (II).

Main Theorem. *Let X be a complete toric variety of dimension n with ample (Cartier) divisor D.*

Received May 10, 2002. Revision received September 9, 2002.
Supported by NSC project: NSC 90-2115-M008-020, Taiwan.

A *The reflexive sheaf $\mathcal{O}(K_X + \ell D)$ is generated by global sections for $\ell \geq n+1$. If X is Gorenstein, then $K_X + nD$ is also generated by global sections unless $(X, D) \cong (\mathbb{P}^n, \mathcal{O}(1))$.*

B *If X is Gorenstein and $\mathbb{Q}$-factorial, then $K_X + \ell D$ is very ample for $\ell \geq n+2$ with $n \leq 6$. For $\ell = n+1$ with $n \leq 4$, this is also true unless $(X, D) \cong (\mathbb{P}^n, \mathcal{O}(1))$. For $n \geq 7$, $K_X + \ell D$ is very ample for $\ell \geq \lceil \frac{3}{2} n \rceil - 1$.*

In Section 2 we review the necessary background in toric geometry. Section 3 contains elementary proofs of the main theorems A and B. An alternative toric proof (modeled on [6]) of Theorem A in the Gorenstein case is given in Section 4, where a toric proof of the singular version of toric Kodaira vanishing theorem is also given. (After completion of this work, I was informed that a proof recently appeared in a preprint by Mustata [13].)

We should remark here that a different toric proof of Theorem A in the Gorenstein case has been found by Laterveer [10] and Fujino [5] using Reid's [15] toric version of Mori theory.

ACKNOWLEDGMENT. In an earlier version (authored with C.-L. Wang, dated October 2000), it was claimed that Theorem B is true without the dimension restriction $n \leq 6$ (cf. Remark 3.2). Upon finding this idea to be mistaken, Wang insisted on his removal as co-author. However, I remain grateful to him for many useful discussions while preparing this note.

2. Review of Toric Geometry

Only necessary material is recalled here, and readers are referred to [2; 7; 9; 14] for details. Let $N \cong \mathbb{Z}^n$ be a lattice with dual $M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. A *cone* $\sigma \subset N_{\mathbb{R}}$ will mean a closed strongly rational polyhedral convex cone with dual $\sigma^\vee \subset M_{\mathbb{R}}$ defined by $\{u \in M_{\mathbb{R}} \mid \langle u, v\rangle \geq 0\ \forall v \in \sigma\}$. Denote by $\partial\sigma$ the collection of cones as faces of σ. A *fan* $\triangle$ of $N_{\mathbb{R}}$ is a collection of cones $\{\sigma\}$ such that (a) if $\tau \in \partial\sigma$ then $\tau \in \triangle$ and (b) if $\sigma_1, \sigma_2 \in \triangle$ then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2. A p-dimensional cone is *simplicial* if it has exactly p edges. A fan is called *complete* if its cones fill up $N_{\mathbb{R}}$. In this paper, we consider mostly complete toric varieties; that is, $\triangle$ is complete. We denote the subset of p-dimensional cones in $\triangle$ by $\triangle_p$.

Fix a ground field k (or in fact we may take $k = \mathbb{Z}$). For a cone $\sigma \subset N_{\mathbb{R}}$, we have that $S_\sigma = \sigma^\vee \cap M$ determines a normal semigroup ring $A_\sigma = k[S_\sigma]$ and an affine toric variety $U_\sigma = \operatorname{Spec} A_\sigma$. The zero cone $0 \in \triangle$ corresponds to the common Zariski open set $U_0 = \operatorname{Spec} k[M] \cong \operatorname{Spec} k[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}] \cong T \cong (k^\times)^n$. For a fan $\triangle$, $X = X(\triangle)$ is the toric variety defined by gluing all the U_σ. Here we associate to each $u \in M$ a monomial x^u, so there is an obvious torus action of T on X. For $\tau \in \triangle_1$, we denote by $\hat{\tau}$ its (integral) primitive generator. Define $\tau^\perp := \{u \in M_{\mathbb{R}} \mid \langle u, \hat{\tau}\rangle = 0\}$. This gives a codimension-1 subtorus $\operatorname{Spec} k[\tau^\perp \cap M]$ of T, and its closure in X in turn gives rise to a T(-invariant) Weil divisor D_τ. On U_σ we have $\operatorname{div} x^u = \sum_{\tau \in \triangle_1 \cap \partial\sigma} \langle u, \hat{\tau}\rangle D_\tau$. More generally, any $w \in \triangle_p$ gives rise to a $(n-p)$ cycle.

Proposition 2.1. *For a T-Weil divisor $D = \sum a_\tau D_\tau$, the following statements hold.*

1. $\Gamma(X, D) = \bigoplus_{u \in P_D \cap M} k \cdot x^u$, *where* $P_D = \{u \in M_{\mathbb{R}} \mid \langle u, \hat{\tau} \rangle \geq -a_\tau \ \forall \tau \in \triangle_1\}$ *is a convex but not necessarily integral polytope.*
2. *On U_σ, the reflexive sheaf $\mathcal{O}(D)$ corresponds to a finitely generated module $A_\sigma\langle x^{m(\sigma)_1}, \dots, x^{m(\sigma)_{r_\sigma}}\rangle$ over A_σ, where a minimal set of generators are assumed to be chosen; then $\mathcal{O}(D)$ is generated by its global sections if and only if (iff) $m(\sigma)_j \in P_D$ for all σ and j.*

A T(-invariant) Cartier divisor D is given by data (U_σ, x^{u_σ}) with $\sigma \in \triangle_n$, $u_\sigma \in M$, and $\langle u_\sigma, \hat{\tau} \rangle = \langle u'_\sigma, \hat{\tau} \rangle$ whenever $\tau \in \triangle_1 \cap \partial\sigma \cap \partial\sigma'$. The associated Weil divisor is given by $\sum_{\tau \in \triangle_1} a_\tau D_\tau$, where $a_\tau = \langle u_\sigma, \hat{\tau} \rangle$ if $\tau \in \triangle_1 \cap \partial\sigma$. In this case, $\Gamma(U_\sigma, D) = A_\sigma\langle x^{-u_\sigma}\rangle$ and $P_D = \bigcap_{\sigma \in \triangle_n}(-u_\sigma + \sigma^\vee)$. Note that T-Cartier divisors are in one-to-one correspondence with $|\triangle|$-supported PL (piecewise linear) functions on $N_{\mathbb{R}}$ that are $\mathbb{Z}$-valued on N. Namely, $h_D(v) = -\langle u_\sigma, v \rangle$ when $v \in \sigma$. Let $\Phi_D \colon X \dashrightarrow \mathbb{P}^{h^0(X,D)-1}$ be the rational map defined by the linear system $|D|$.

Proposition 2.2. *Let D be a T-Cartier divisor. Then:*

1. *$\mathcal{O}(D)$ is generated by global sections iff Φ_D is a morphism—that is, $|D|$ is basepoint-free iff P_D is an integral polytope with vertexes $\{-u_\sigma \mid \sigma \in \triangle_n\}$ (possibly with repetition) iff h_D is convex;*
2. *D is very ample (that is, Φ_D is a closed embedding) iff, for all $\sigma \in \triangle_n$, $u_\sigma + P_D \cap M$ generates $\sigma^\vee \cap M$ as a semigroup; and*
3. *D is ample (that is, ℓD is very ample for ℓ large) iff P_D is an integral polytope with vertexes $\{-u_\sigma \mid \sigma \in \triangle_n\}$ (without repetition) iff h_D is strictly convex.*

Corollary 2.3. *For complete toric varieties, ample divisors are generated by global sections. In fact, D is ample iff Φ_D is a finite morphism.*

Toric varieties are naturally Cohen–Macaulay (they have only rational singularities), with canonical (T-Weil) divisor $K = -\sum_{\tau \in \triangle_1} D_\tau$. Hence X is Gorenstein (resp., $\mathbb{Q}$-Gorenstein of index r) iff K (resp., rK) is Cartier; that is, K is given by data $\{k_\sigma \in M \mid \sigma \in \triangle_n\}$ such that $\langle k_\sigma, \hat{\tau} \rangle = -1$ (resp., $-r$) for $\tau \in \triangle_1 \cap \partial\sigma$. We also have that X is $\mathbb{Q}$-factorial iff $\triangle$ is simplicial (i.e., consists of simplicial cones) and that X is factorial iff the set of primitive generators of edges of each cone is part of a $\mathbb{Z}$-basis of M iff X is nonsingular.

Theorem 2.4. *Let D be an ample divisor.*

1. (Demazure) *If X is nonsingular then D is very ample.*
2. (Ewald–Wessels [4]) *For* $\dim X = n \geq 2$, *$(n-1)D$ is very ample.*

3. Proof of the Main Theorem

We start with the following trivial but important observation. If $W = \sum a_\tau D_\tau$ is a Weil divisor, then

$$\begin{aligned}\operatorname{Int} P_W \cap M &= \{u \in M \mid \langle u, \hat{\tau}\rangle + a_\tau > 0\ \forall \tau \in \triangle_1\} \\ &= \{u \in M \mid \langle u, \hat{\tau}\rangle + a_\tau - 1 \geq 0\ \forall \tau \in \triangle_1\} \\ &= P_{W+K} \cap M.\end{aligned}$$

These equalities hold also for incomplete toric varieties (e.g., U_σ).

Proof of Theorem A

Now let D be an ample divisor given by the local data (U_σ, x^{u_σ}). Applying the previous argument to $W = \ell D$ yields $P_{\ell D+K} \cap M = \operatorname{Int} P_{\ell D} \cap M$.

For each $\sigma \in \triangle_n$, if we apply the foregoing to U_σ with $W = 0$ then we obtain the canonical module $\Gamma(U_\sigma, \mathcal{O}(K)) = A_\sigma\langle x^{m(\sigma)_1}, \dots, x^{m(\sigma)_r}\rangle$ with exponents consisting of $\operatorname{Int}\sigma^\vee \cap M$. Here we may (and do) choose $\{m(\sigma)_j\}_{j=1,\dots,r}$ to be a minimal generating set that lies in the "quasi-box"

$$B_{(0,1]} = \left\{\sum_{i=1}^{n} a_i v_i \mid 0 < a_i \leq 1,\ v_1, \dots, v_n : n \text{ distinct primitive generators of (one-dimensional) edges of } \sigma^\vee\right\}$$

($B_{[0,1]}$ is defined similarly). Then $\{-\ell u_\sigma + m(\sigma)_j\}_{j=1,\dots,r}$ is a minimal generating set of $\Gamma(U_\sigma, \mathcal{O}(\ell D + K))$. In order to show that $\Gamma(X, \mathcal{O}(\ell D + K))$ generates $\mathcal{O}(\ell D + K)$ on U_σ, we must show that $P_{\ell D+K}$ contains the $(-\ell u_\sigma + m(\sigma)_j)$—in other words, that $\ell u_\sigma + \operatorname{Int} P_{\ell D} = \operatorname{Int}(\ell(u_\sigma + P_D))$ contains the $m(\sigma)_j$.

Let us define the "quasi-simplex" $S_{[a,b]}$ for $a \leq b$ to be the part of $\sigma^\vee$ that has the form $\sum a_i v_i$ with $0 \leq a_i$ and $a \leq \sum a_i \leq b$, where the v_i are among the primitive generators of edges of $\sigma^\vee$ ($S_{(0,c)}$, $S_{(0,c]}$, etc. are defined similarly). The point is that, since D is ample, $c(u_\sigma + P_D) \supset S_{[0,c]}$ and $\operatorname{Int} c(u_\sigma + P_D) \supset S_{(0,c)}$ for all $c > 0$. Hence

$$\operatorname{Int}((n+1)(u_\sigma + P_D)) \supset S_{(0,n+1)} \supset B_{(0,1]} \supset \{m(\sigma)_j\}_{j=1,\dots,r}.$$

This proves that $\mathcal{O}((n+1)D + K)$ is generated by global sections.

When X is Gorenstein, $\Gamma(U_\sigma, \mathcal{O}(K)) = A_\sigma\langle x^{-k_\sigma}\rangle$ with $\operatorname{Int}\sigma^\vee \cap M = -k_\sigma + \sigma^\vee \cap M$. For $nD + K$ to be generated by global sections is equivalent to having, for all $\sigma \in \triangle_n$, that $-k_\sigma \in \operatorname{Int} n(u_\sigma + P_D)$. However, $\operatorname{Int}(n(u_\sigma + P_D)) \supset S_{(0,n)} \supset B_{(0,1)}$ and $B_{(0,1]} \ni -k_\sigma$ and so it follows that, if there is a σ with $-k_\sigma \notin \operatorname{Int} n(u_\sigma + P_D)$, then $-k_\sigma$ must lie in the boundary of $n(u_\sigma + P_D)$ and be of the form $-k_\sigma = v_1 + \dots + v_n$, where the v_i are distinct primitive generators of $\sigma^\vee$. Moreover, there are no lattice points in $\operatorname{Int} B_{[0,1)}$ (actually, no lattice points in $\operatorname{Int} n(u_\sigma + P_D)$) nor any lattice points in lower faces of $B_{[0,1]}$, except the vertices. This implies that $v_1, \dots, v_n$ form a $\mathbb{Z}$-basis of M.

We claim that there is no other primitive generator v of σ. If such a v did exist, then $\operatorname{Int} B_{[0,1)} \cap M = \emptyset$ implies that any $n-1$ v_i, together with v, would still form a $\mathbb{Z}$-basis of M. Write $v = \sum_{i=1}^{n} c_i v_i$. Computation of determinant shows that $|c_i| = 1$ for all i. Since v is also a primitive generator, at least one c_i must be -1. Let I be the subindex set of $\{1, \dots, n\}$ such that $c_i = -1$ for all $i \in I$ and $c_j = 1$

for all $j \notin I$. Consider the element $w = v + \sum_{i \in I} v_i = \sum_{j \notin I} v_j$. Then $w = \frac{1}{2}\big(v + \sum_{i=1}^{n} v_i\big)$ would be a nontrivial interior lattice point of $\operatorname{Int} n(u_\sigma + P_D)$, a contradiction.

It follows that, if $nD + K$ is not generated by global sections on some U_σ, then $n(u_\sigma + P_D) = S_{[0,n]}$ and the polytope $u_\sigma + P_D = S_{[0,1]}$ is the regular n-simplex, with $v_1, \dots, v_n$ the edges through 0. Because D is ample, this implies that $(X, D) \cong (\mathbb{P}^n, \mathcal{O}(1))$. □

Proof of Theorem B

The $\mathbb{Q}$-factorial assumption asserts that all cones involved are simplicial. Let us first suppose that X is singular on some open set U_σ with $\sigma \in \triangle_n$. Since X is Gorenstein, this is equivalent (by our previous argument) to $-k_\sigma \neq v_1 + \cdots + v_n$ for v_i the primitive generators of edges of $\sigma^\vee$.

Claim 3.1. *If* $\ell \geq \max\big\{n+1, \big\lceil \frac{3}{2}n \big\rceil - 1\big\}$, *then* $\ell D + K$ *is very ample on* U_σ. *That is,* $\Phi_{\ell D+K}|_{U_\sigma}$ *is a closed embedding.*

Proof. Let $B_{[0,1]}, S_{(a,b)}, S_{[a,b)}, \dots$ be the same subsets of $\sigma^\vee$ as before. We set also $S_c = S_{[c,c]}$. There is an obvious reflection of lattice points in $B_{[0,1]}$ with respect to the center $\frac{1}{2}\sum v_i$, namely $B_{[0,1]} \cap M \ni \alpha \mapsto \alpha' = \sum v_i - \alpha \in B_{[0,1]} \cap M$. This reflects $S_{[a,b)}$ to $S_{(n-b,n-a]}$.

By assumption we have that $-k_\sigma \in S_\lambda$ for $0 < \lambda < n$ ($\lambda \in \mathbb{Q}$) and there is no interior lattice point in $S_{[0,\lambda)}$. By reflection, there is also no interior lattice point in $S_{(n-\lambda,n]} \cap B_{[0,1]}$.

If $-k_\sigma \in \operatorname{Int} B_{[0,1]}$ then $n - \lambda \geq \lambda$ (i.e., $\lambda \leq n/2$). In general, let $-k_\sigma = \sum_{i=1}^{n} a_i v_i$. By reordering the v_i if necessary, we may assume that there exists an $m \in \mathbb{N}$ with $0 < a_i < 1$ for $i \leq m$ and with $a_i = 1$ for $i \geq m+1$. If $m < n$ then $-k_\sigma$ is in the interior of an "upper face" F'_m of $B_{[0,1]}$. In this case $-k' := \sum_{i=1}^{m} a_i v_i$ is an interior lattice point of the "lower face" F_m of $B_{[0,1]}$, which makes the cone $\sigma^{\vee\prime}$ spanned by $v_1, \dots, v_m$ Gorenstein because $\{v_{m+1}, \dots, v_n\}$ and $-k'$ generate $-k_\sigma$:

$$\begin{aligned} m \in \operatorname{Int} \sigma^{\vee\prime} \cap M &\implies m + v_{m+1} + \cdots + v_n \in \operatorname{Int} \sigma^\vee \cap M \\ &\implies m + v_{m+1} + \cdots + v_n = -k_\sigma + \gamma, \quad \gamma \in \sigma^\vee \cap M \\ &\implies m = -k' + \gamma \quad (\text{and so } \gamma \in \sigma^{\vee\prime} \cap M). \end{aligned}$$

Observe also that when $m < n$ there are no interior lattice points of $B_{[0,1]}$ and $\{v_{m+1}, \dots, v_n\}$ is a $\mathbb{Z}$-basis of $\mathbb{R}\langle v_{m+1}, \dots, v_n\rangle \cap M$. Moreover, $B_{[0,1]} \cap M$ is generated by $\{v_{m+1}, \dots, v_n\} \subset S_1$ and $F_m \cap M$, on which $-k' \in S_{\lambda'}$ with $\lambda' \leq m/2$ and $\lambda = \lambda' + (n-m) \leq n - m/2$.

Let $\beta \in B_{[0,1]} \cap M$. If $\beta \in \operatorname{Int} \sigma^\vee$, the Gorenstein property implies that $\beta = -k_\sigma + \gamma$ for some $\gamma \in \sigma^\vee \cap M$ and $\gamma \in S_{[0,n-\lambda]}$. If $\beta \notin \operatorname{Int} \sigma^\vee$, then we may assume modulo S_1 that $\beta \in \operatorname{Int} F_m$ (for $m = n$, $F_m := B_{[0,1]}$). The result of Ewald and Wessels [4] states that β is generated by $S_{[0,m-1]} \cap F_m \cap M$ (cf. Theorem 2.4(2) and Remark 3.2).

The "very ampleness" of $\ell D + K$ on U_σ is equivalent to the fact that $(\ell u_\sigma + k_\sigma) + P_{\ell D+K} \cap M$ generates $B_{[0,1]} \cap M$, since the latter generates $\sigma^\vee \cap M$ via translations. Given

$$\begin{aligned}(\ell u_\sigma + k_\sigma) + P_{\ell D+K} \cap M &= k_\sigma + \ell u_\sigma + \operatorname{Int} P_{\ell D} \cap M \\ &= k_\sigma + \operatorname{Int}(\ell(u_\sigma + P_D)) \cap M \\ &\supset k_\sigma + S_{(0,\ell)} \cap M \supset S_{[0,\ell-\lambda)} \cap M,\end{aligned}$$

we need only require that $\ell - \lambda > \max\{n - \lambda, m - 1, \lambda', 1\}$. That is, $\ell > \max\{n, \lambda + m - 1, \lambda + \lambda', \lambda + 1\}$. Now

$$\begin{aligned}\lambda + m - 1 &< n + m/2 - 1 \le \tfrac{3}{2}n - 1, \\ \lambda + \lambda' &= 2\lambda' + (n - m) \le n, \\ \lambda + 1 &< n - m/2 + 1 \le n + 1/2.\end{aligned}$$

The claim is proved. □

Notice that $\lceil \frac{3}{2}n \rceil - 1 \le n + 2$ for $n \le 6$ and $\lceil \frac{3}{2}n \rceil - 1 \le n + 1$ for $n \le 4$. In the range $n \le 4$, $K_X + (n+1)L$ fails to be very ample only if X is nonsingular on some U_σ, hence $-k_\sigma = \sum v_i \in S_n$. Since the v_i already form a $\mathbb{Z}$-basis of M and

$$(n+1)u_\sigma + k_\sigma + P_{(n+1)D+K} \cap M \supset k_\sigma + S_{(0,n+1)} \cap M \supset S_{[0,1)} \cap M,$$

it is clear that $(\ell u_\sigma + k_\sigma) + P_{\ell D+K} \cap M \supset S_{[0,1]} \cap M$ for all $\ell > n + 1$ and that $\ell D + K$ is very ample on U_σ for $\ell \ge n+2$. (In particular, this gives a simple toric proof of Fujita's conjecture for nonsingular toric varieties in any dimensions.)

Moreover, if $(n+1)D + K$ fails to be very ample on U_σ, then $(n+1)u_\sigma + k_\sigma + P_{(n+1)D+K} \cap M \not\supset S_{[0,1]} \cap M$. That is, $v_i \notin (n+1)u_\sigma + k_\sigma + P_{(n+1)D+K} \cap M$ for some (in fact, all) i. This implies that $(n+1)(u_\sigma + P_D) = S_{[0,n+1]}$. Indeed, if $(n+1)(u_\sigma + P_D)$ properly contains $S_{[0,n+1]}$ then

$$\begin{aligned}v_i + (-k_\sigma) &= v_i + (v_1 + v_2 + \cdots + v_n) \in S_{n+1} \cap \operatorname{Int} \sigma^\vee \cap M \\ &\subset \operatorname{Int}(n+1)(u_\sigma + P_D) \cap M = (n+1)u_\sigma + P_{(n+1)D+K} \cap M,\end{aligned}$$

which is a contradiction!

Therefore, the polytope $u_\sigma + P_D$ must be the regular n-simplex with $v_1, \ldots, v_n$ the edges through 0. Since D is ample, this implies that $(X, D) \cong (\mathbb{P}^n, \mathcal{O}(1))$. □

REMARK 3.2. Wang has conjectured that, for an n-dimensional Gorenstein cone $\sigma^\vee$ with $-k_\sigma \in \operatorname{Int} B_{[0,1]}$, $B_{[0,1]}$ is generated by $S_{[0,n/2]} \cap M$. If this is true then the foregoing argument will lead to a proof of conjecture (II) in the singular case in any dimension. No counterexample has been found in a Maple program search.

4. Toric Vanishing Theorems

The following Kodaira-type vanishing theorem (Theorem 4.1(2)) for ample line bundles on toric varieties was stated without proof in [2, (7.5.2)] and [14, p. 130].

For the reader's convenience we give a proof here assuming only that the bundle is big and nef (Kawamata–Viehweg vanishing theorem). Part 1 is a more standard fact in toric geometry. We put them together not only for completeness but also because their proofs are along the same line.

THEOREM 4.1. *Let X be a complete toric variety, and let D be a Cartier divisor that is generated by global sections.*

1. $H^i(X, D) = 0$ *for* $i \geq 1$.
2. *If, moreover, D is big (e.g., ample), then* $H^i(X, K + D) = 0$ *for* $i \geq 1$.

Proof. By Demazure's graded decomposition theorem for the Cartier divisor L with associated PL function h (see [2, 7.2; 9, p. 42]),

$$H^i(X, L) = \bigoplus_{m \in M} H^i_{Z(m,h)}(N_{\mathbb{R}}, k),$$

where $Z(m, h) = \{n \in N_{\mathbb{R}} \mid \langle m, n\rangle \geq h(n)\}$. Moreover, for $i \geq 2$ we have

$$H^{i-1}(N_{\mathbb{R}} - Z(m, h), k) \cong H^i_{Z(m,h)}(N_{\mathbb{R}}, k),$$

and then there exists an exact sequence

$$0 \to H^0_{Z(m,h)}(N_{\mathbb{R}}, k) \to k \to H^0(N_{\mathbb{R}} - Z(m, h), k) \to H^1_{Z(m,h)}(N_{\mathbb{R}}, k) \to 0.$$

Note that $Z(m, h) = N_{\mathbb{R}}$ if and only if x^m is a section of L.

For the proof of part 1, let $L = D$. Since h is convex (D is generated by global sections), $N_{\mathbb{R}} - Z(m, h) = \{n \in N_{\mathbb{R}} \mid \langle m, n\rangle < h(n)\}$, which is a convex open cone (hence, contractible) and so $H^i_{Z(m,h)}(N_{\mathbb{R}}, k) = 0$ for $i \geq 2$. To achieve the desired vanishing for $i = 1$, if x^m is a section of D then $Z(m, h) = N_{\mathbb{R}}$ and $H^0(N_{\mathbb{R}} - Z(m, h), k) = H^0(\emptyset, k) = 0$, so $H^1_{Z(m,h)}(N_{\mathbb{R}}, k) = 0$. And if x^m is not a section of D then $Z(m, h) \neq N_{\mathbb{R}}$. By the definition of local cohomology, we have $H^0_{Z(m,h)}(N_{\mathbb{R}}, k) = 0$. The previously displayed exact sequence again implies that $H^1_{Z(m,h)}(N_{\mathbb{R}}, k) = 0$.

Part 2, by a Grothendieck–Serre duality theorem for Cohen–Macaulay schemes (see [2, (7.7.1)] for a toric proof in the toric case), is equivalent to $H^i(X, -D) = 0$ for all $i \leq n - 1$. Let $L = -D$. Then $h = h_{-D} = -h_D$ is concave, so $Z(m, h)$ is a closed convex cone. In this case $-D$ has no sections, so $Z(m, h) \neq N_{\mathbb{R}}$ for all m. By an argument similar to our proof of part 1, we have $H^0_{Z(m,h)}(N_{\mathbb{R}}, k) = 0$ and $H^1_{Z(m,h)}(N_{\mathbb{R}}, k) = 0$.

For other i, since D is assumed to be big, it follows by Lemma 4.2 that $Z(m, h)$ cannot contain any positive-dimensional vector subspace of $N_{\mathbb{R}}$: if for some $n \neq 0$ we have $n \in Z(m, h)$ and $-n \in Z(m, h)$, then by adding together $h_D(n) + \langle m, n\rangle \geq 0$ and $h_D(-n) + \langle m, -n\rangle \geq 0$ we obtain (by convexity of h_D) that $0 = h_D(n + (-n)) \geq h_D(n) + h_D(-n) \geq 0$. That is, $h_D(-n) = -h_D(n)$ and so $h_D|_{\mathbb{R}^n}$ is linear—a contradiction. Notice that $Z(m, h)$ may consist of just a single point 0. Now it is easy to see that $H^j(N_{\mathbb{R}} - Z(m, h), k) = 0$ for $1 \leq j \leq n - 2$, since $N_{\mathbb{R}} - Z(m, h)$ is either contractible or homotopic equivalent to the $(n - 1)$-dimensional unit sphere. The proof is complete. □

LEMMA 4.2. *Let D be a Cartier divisor that is generated by global sections. Let d be the dimension of the maximal vector subspace V of $N_{\mathbb{R}}$ such that $h_D|_V$ is linear, and let Φ_D be the projective morphism defined by $|D|$. Then*

$$\dim \text{Image } \Phi_D = \dim P_D = n - d.$$

Proof. The first equality follows from basic properties of Kodaira dimension. For the second equality, recall that $P_D = \{u \in M_{\mathbb{R}} \mid u \geq h_D\}$ by Proposition 2.1(1) and the definition of h_D. If the v_i are a basis of V, then $u \in P_D$ implies that $u(v_i) \geq h_D(v_i)$ and $-u(v_i) = u(-v_i) \geq h_D(-v_i) = -h_D(v_i)$. That is, $u|_V = h_D|_V$. Hence the degree of freedom of u is $n - d$. □

One also has a nice understanding of $H^0(X, K + D)$ by the following lemma.

LEMMA 4.3 [7, p. 90]. *Let X be a complete Gorenstein toric variety and D an ample* (*Cartier*) *divisor. If $\Gamma(X, K + D) \neq 0$ then $K + D$ is generated by global sections. In fact, P_{K+D} is the convex hull of* $\operatorname{Int} P_D \cap M$.

EXAMPLE 4.4. The conclusion of Lemma 4.3 is wrong if X is only $\mathbb{Q}$-Gorenstein. Let $M \cong \mathbb{Z}^4$ with $v_1 = (1, 0, 0, 0)$, $v_2 = (0, 1, 0, 0)$, $v_3 = (0, 0, 1, 0)$, $v_4 = (1, 1, 1, 3)$, and P the convex hull of $\langle 0, v_1, v_2, v_3, v_4 \rangle$. Now P determines a $\mathbb{Q}$-factorial toric variety $X(\triangle)$ and an ample divisor D such that $P_D = P$. Let $\sigma^\vee$ be the cone spanned by v_1, v_2, v_3, v_4 (and hence $u_\sigma = 0$). Then the canonical module $\Gamma(U_\sigma, \mathcal{O}(K)) = A_\sigma \langle x^{m_1}, x^{m_2} \rangle$, with $m_1 = (1, 1, 1, 1)$ and $m_2 = (1, 1, 1, 2)$; in fact, $\operatorname{Int} B_{[0,1]} \cap M = \{m_1, m_2\}$. It is easily seen that $P_{2D+K} \cap M = \operatorname{Int} P_{2D} \cap M = \operatorname{Int} 2P \cap M$, which contains m_2 but not m_1. So $\Gamma(X, K + 2D) \neq 0$, but the reflexive sheaf $\mathcal{O}(K + 2D)$ is not generated by its global sections on U_σ.

This example is inspired by the work of Ewald and Wessels [4]. It can easily be generalized to higher dimensions.

Alternative Proof of Theorem A in the Gorenstein Case

By Lemma 4.3 we need only show that $K + \ell D$ has a nontrivial section for some $\ell \leq n+1$. The Euler characteristic $p(\ell) := \chi(X, K + \ell D)$ is a polynomial in ℓ of degree $\leq n$ and in the range $\ell \in \mathbb{N}$, $p(\ell) = h^0(X, K + \ell D)$, by Theorem 4.1(2). If $K + \ell D$ has no sections for $1 \leq \ell \leq n$, then $p(\ell)$ has roots $1, \ldots, n$ and hence $p(n + 1) \neq 0$, because p is a nontrivial polynomial.

If $K + nD$ is not generated by global sections, then $p(\ell)$ has roots $1, \ldots, n$. Therefore,

$$p(\ell) = \chi(X, K + \ell D) = c(\ell - 1) \cdots (\ell - n).$$

Using the formula given by the Riemann–Roch theorem for line bundles on possibly singular toric varieties,

$$p(\ell) = (-1)^n \chi(X, -\ell D) = \left[e^{-\ell D} \cdot \left(1 - \frac{K}{2} + \cdots \right) \right]_{(n)}$$

$$= \frac{D^n}{n!} \ell^n + \frac{D^{n-1}K}{2(n-1)!} \ell^{n-1} + O(\ell^{n-1})$$

(see [7, Sec. 5.3]), we get that $c = D^n/n!$ and $D^{n-1}K = -(n+1)D^n$. That is,

$$(K + (n+1)D) \cdot D^{n-1} = 0.$$

Because $K+(n+1)D$ is effective and D is ample, this implies that $K+(n+1)D = 0$. But then $D^n = p(n+1) = h^0(X, K + (n+1)D) = h^0(X, \mathcal{O}) = 1$, so $h^0(X, D) = h^0(X, K + (n+2)D) = p(n+2) = n+1$. Consider the projective morphism $\Phi_D \colon X \to \mathbb{P}^n$ defined by $|D|$ with $D = \phi^*H$, where H is the hyperplane class. It is, in general, a finite morphism by Corollary 2.3. Moreover, Φ_D is also a birational morphism (of degree 1) onto $\mathbb{P}^n$ since $D^n = 1$. Hence Φ_D is an isomorphism and $(X, D) \cong (\mathbb{P}^n, \mathcal{O}(1))$. □

Remark 4.5. The idea of this proof follows Fujita's paper [6] closely. It uses the Riemann–Roch theorem and so is not as elementary as the previous proof in Section 3. My motivation for giving this proof is to demonstrate a special feature of toric varieties.

Remark 4.6. Theorem A in the Gorenstein case has been proved by Laterveer [10] using different methods. In [10] it is also claimed that $K+(n+2)D$ is very ample. However, there is a mistake in [10, p. 457]: If we replace t, L, and X by $n+2$, $\mathcal{O}(1)$, and $\mathbb{P}^n$ (respectively) then we get a contradiction to his claim that the rational polyhedron $P_{K_X+tL} = P_L$ contains the rational polyhedron $P_{(t-1)L} = P_{(n+1)L}$. The correct version of this inclusion is $P_{K_X+tL} \cap M = \operatorname{Int} P_{tL} \cap M$.

Question 4.7. In the second part of Theorem A, can one relax the assumption on X to be $\mathbb{Q}$-Gorenstein or perhaps even all the assumptions? Also, can one remove the $\mathbb{Q}$-factoriality assumption on X in Theorem B?

5. Appendix: Toric Nakai–Moishezon–Kleiman Criterion

Results in this section are well known to experts and are essentially contained in [14; 15], though not stated in generality here. Because they are crucial for us to fix ideas when working on toric varieties, we give the proofs for the reader's convenience. (In fact, the result in this appendix has already appeared in [12]; however, it is hoped that the treatment here has some independent interest.)

Assume first that $\triangle$ is a complete simplicial fan of dimension n and that D is a T-invariant Cartier divisor with data (U_σ, x^{u_σ}). Let $\omega \in \triangle_{n-1}$ and let l_ω be the corresponding 1-cycle as in Section 2. Suppose that ω separates two cones σ and σ' in $\triangle_n$. Let $e_1, \dots, e_{n-1}$ be the primitive generators of edges of ω, and let e_n and e_{n+1} be the primitive generators of opposite edges of σ and σ', respectively. Because $e_1, \dots, e_n$ form a $\mathbb{Q}$-basis of N, we have the relation $\sum_{i=1}^{n+1} a_i e_i = 0$ with $a_{n+1} = 1$ and $a_n > 0$. Recall now the following formulas from [15, (2.7)]:

(1) $D_e l_\omega = 0$ if $e \notin \{e_1, \dots, e_{n+1}\}$,
(2) $D_{e_i} l_\omega = a_i D_{e_{n+1}} l_\omega$ for $i = 1, \dots, n$, and
(3) $D_{e_{n+1}} l_\omega = \operatorname{mult}(\omega) \operatorname{mult}(\sigma') > 0$,

where $\operatorname{mult}(\omega) = [N_\omega : \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_{n-1}]$ and N_ω is the sublattice of N generated (as a group) by $\omega \cap N$, and similarly for $\operatorname{mult}(\sigma')$ (see [7, p. 100]).

LEMMA 5.1. $Dl_\omega = \langle u_{\sigma'} - u_\sigma, e_{n+1}\rangle D_{e_{n+1}} l_\omega$.

Proof. By formula (1), $Dl_\omega = \sum_{i=1}^{n+1} d_{e_i} D_{e_i} l_\omega$, which equals $\left(\sum_{i=1}^{n+1} d_{e_i} a_i\right) D_{e_{n+1}} l_\omega$ by (2). For $i = 1, \dots, n$ we have $d_{e_i} = \langle u_\sigma, e_i\rangle$, so

$$\sum_{i=1}^{n+1} a_i d_{e_i} = \left\langle u_\sigma, \sum_{i=1}^{n} a_i e_i \right\rangle + d_{e_{n+1}} a_{n+1}$$
$$= \langle u_\sigma, -e_{n+1}\rangle + \langle u_{\sigma'}, e_{n+1}\rangle = \langle u_{\sigma'} - u_\sigma, e_{n+1}\rangle$$

and then $Dl_\omega = \langle u_{\sigma'} - u_\sigma, e_{n+1}\rangle D_{e_{n+1}} l_\omega$. □

PROPOSITION 5.2. *Let h_D be the PL function defined by D. Then*

1. *h_D is convex on $\sigma \cup \sigma'$ iff $Dl_\omega \geq 0$, and*
2. *h_D is strictly convex on $\sigma \cup \sigma'$ iff $Dl_\omega > 0$.*

Proof. Notice that h_D is convex iff $h_D(w) \leq -\langle u_\sigma, w\rangle$ for all $w \in \triangle_1$ and $\sigma \in \triangle_n$. That is, $\langle u_{\sigma'} - u_\sigma, w\rangle \geq 0$ for all $w \in \sigma'$. By Lemma 5.1 and formula (3), this is equivalent to $Dl_w \geq 0$. The strictly convex case is entirely similar. □

THEOREM 5.3 (Toric Nakai–Moishezon–Kleiman criterion). *For any complete toric variety X with D a Cartier divisor*:

1. *D is generated by global sections iff D is nef*; *and*
2. *D is ample iff D is numerically positive.*

Proof. If the fan is simplicial then this follows from Proposition 2.2 and Proposition 5.2. In the general case, part 1 again follows from the simplicial case: consider subdivision of $\triangle$ into the simplicial fan $\triangle'$ and let $\phi\colon X' = X'(\triangle') \to X = X(\triangle)$ be the corresponding toric birational morphism. Then notice that D is nef on X iff ϕ^*D is nef on X' and that ϕ^*D is generated by global sections on X' iff D is generated by global sections on X.

Part 2 follows from part 1: D is ample certainly implies that it has positive degree when restricted to any effective curve; conversely, if D is numerically positive then by part 1 $|D|$ defines a morphism Φ_D, which has no positive-dimensional fiber because otherwise D would have zero degree along curves in the fiber. Hence Φ_D is finite, and this implies that D is ample. □

Added in proof. Sam Payne has informed the author that Lemma 4.3 quoted from [7], on which our alternative proof of Theorem A is based, does not seem to have a known valid proof.

References

[1] U. Angerhn and Y.-T. Siu, *Effective freeness and point separation for adjoint bundles,* Invent. Math. 122 (1995), 291–308.

[2] V. Danilov, *The geometry of toric varieties,* Russian Math. Surveys 33 (1978), 97–154.

[3] L. Ein and R. Lazarsfeld, *Global generation of pluricanonical and adjoint linear series on smooth projective threefolds,* J. Amer. Math. Soc. 6 (1993), 875–903.

[4] G. Ewald and U. Wessels, *On the ampleness of invertible sheaves in complete projective toric varieties,* Results Math. 19 (1991), 275–278.
[5] O. Fujino, *Notes on toric varieties from Mori theoretic viewpoint,* preprint, math.AG/0112090.
[6] T. Fujita, *On polarized manifolds whose adjoint bundles are not semipositive,* Algebraic geometry (Sendai, Japan, 1985), pp. 167–178, North-Holland, Amsterdam, 1987.
[7] W. Fulton, *Introduction to toric varieties,* Princeton Univ. Press, Princeton, NJ, 1993.
[8] Y. Kawamata, *On Fujita's freeness conjecture for 3-folds and 4-folds,* Math. Ann. 308 (1997), 491–505.
[9] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings I,* Lecture Notes in Math., 339, Springer-Verlag, Berlin, 1973.
[10] R. Laterveer, *Linear systems on toric varieties,* Tôhoku Math. J. (2) 48 (1996), 451–458.
[11] H.-W. Lin, *Adjoint linear systems and Fujita's conjecture on toric varieties,* Ph.D. thesis, National Taiwan Normal University, 1998.
[12] A. R. Mavlyutov, *Semiample hypersurfaces in toric varieties,* Duke Math. J. 101 (2000), 85–116.
[13] M. Mustata, *Vanishing theorems on toric varieties,* Tôhoku Math. J. (2) 54 (2002), 451–470.
[14] T. Oda, *Convex bodies and algebraic geometry,* Springer-Verlag, New York, 1988.
[15] M. Reid, *Decomposition of toric morphisms,* Arithmetic and geometry (M. Artin, J. Tate, eds.), Progr. Math., 39, pp. 395–418, Birkhäuser, Boston, 1983.
[16] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces,* Ann. of Math. (2) 127 (1988), 309–316.
[17] K. Smith, *Fujita's freeness conjecture in terms of local cohomology,* J. Algebraic Geom. 6 (1997), 417–429.

Department of Mathematics
National Central University
Chung-Li
Taiwan

linhw@math.ncu.edu.tw