ICCM 2007 · Vol. II · 1–4

Quantum Invariance of Simple Flops

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Abstract

This note is a supplementary reading for the joint paper [8] with Yuan-Pin Lee and Chin-Lung Wang, entitled "Flops, motives and invariance of quantum rings". In this note, I report our main result by a conceptual description instead of giving logically strict proofs. About the degeneration part which consists of complicated induction procedures, I provide some examples to illustrate it. I hope that readers can catch the key idea of our paper quickly through this note.

2000 Mathematics Subject Classification: 14N35, 14E30.
Keywords and Phrases: Ordinary flops, Chow motives, quantum cohomology, analytic continuations, Kähler moduli space.

1. Motivation

If not specifically stated, the ground field is assumed to be the complex numbers $\mathbb{C}$. Two algebraic varieties are called birational if they have an isomorphic Zariski open subset. The problem of classifying varieties up to birational equivalence is usually the main interest of algebraic geometers. One of the main goals in birational geometry is to find a good geometric model that is convenient for the study of the given algebraic variety or its function field.

For 1-dimensional case, there is a unique nonsingular projective curve in a fixed birational equivalence class. For 2-dimensional case, there are possibly many smooth surfaces in a fixed birational equivalence class. At the beginning of the 20th century (c.f. [2]), Italian algebraic geometers applied the Castelnuovo's contraction theorem to a smooth surface X repeatedly to obtain a *minimal surface* which contains no (-1) rational curve. When $\kappa(X) = -\infty$, that is $\Gamma(X, K_X^m) = 0$ for all $m \in \mathbb{N}$,

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Enrique's theorem says that X is birational to a ruled surface $C \times \mathbb{P}^1$. When $\kappa(X) \geq 0$, that is $\Gamma(X, K_X^m) \neq 0$ for some $m \in \mathbb{N}$, the minimal model is unique.

In 1982, Mori proved the three dimensional generalization of Castelnuovo's contraction theorem to continue the *minimal model program.* The existence of minimal models has later been achieved in dimensions three and four (c.f. [6] and the references therein). However, since minimal models in a fixed birational equivalence class are in general not unique, one of the most important problem remaining is to find invariants among birational minimal models.

For this purpose, C.-L. Wang raised the notion of K-equivalent varieties to generalize the one of minimal models [19]. Two ($\mathbb{Q}$-Gorenstein) varieties X and X' are *K-equivalent* if there exist birational morphisms $\phi : Y \to X$ and $\phi' : Y \to X'$ with Y smooth

$$\begin{tikzcd} & Y \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\ X & & X' \end{tikzcd}$$

such that

$$\phi^* K_X = {\phi'}^* K_{X'}.$$

Two birational minimal models are automatically K-equivalent, so we turn our attention to study K-equivalent varieties.

V. Batyrev [1] and C.-L. Wang [19] showed that K-equivalent smooth varieties have the same Betti numbers. However, the cohomology ring structures are in general different. Two natural questions arise here:

1. Is there a *canonical* correspondence between the cohomology groups of K-equivalent smooth varieties?
2. Is there a modified ring structure which is invariant under the K-equivalence relation?

The following conjecture was advanced by Y. Ruan [18] and C.-L. Wang [20] in response to these questions.

Conjecture 1 *K-equivalent smooth varieties have canonically isomorphic quantum cohomology rings over the extended Kähler moduli spaces.*

For threefolds this Conjecture was proved by A. Li and Y. Ruan [10]. Our work is to study this conjecture in higher dimensional case. In dimension three, by generalizing Kollár's result [7], any K-equivalent map can be connected by a finite sequence of algebraic surgeries called *flops.* Roughly, each flop is obtained by removing one chain of rational curves C in X with $K_X|_C = 0$ then gluing back C into the open space

$X \backslash C$ in a different manner. Among them, the Atiyah $\mathbb{P}^1$ flop is the simplest one.

For higher dimensions, the natural generalizations of the Atiyah flop are called *ordinary $\mathbb{P}^r$ flops*. Ordinary flops are not only of the simplest type, but also crucial to the general theory of minimal models and K equivalence, so we make the choice to start with them.

Our main results in [8] answer the first question for general ordinary $\mathbb{P}^r$ flops and the second question for simple $\mathbb{P}^r$ flops.

Theorem 2 *For ordinary flops, the correspondence defined by the graph closure gives equivalence of Chow motives and preserves the Poincaré pairing.*

While the ring structure is in general not preserved under this correspondence, the quantum cohomology ring is, when the analytic continuation on the Novikov variables is allowed.

Theorem 3 *The big quantum cohomology ring is invariant under simple ordinary flops, after an analytic continuation over the extended Kähler moduli space.*

2. Cohomology correspondence

2.1. Ordinary $\mathbb{P}^r$ flops.

Let X be a smooth complex projective manifold and $\psi : X \to \bar{X}$ a flopping contraction in the sense of minimal model theory, with $\bar{\psi} : Z \to S$ the restriction map on the exceptional loci. Assume that

(i) $\bar{\psi}$ equips Z with a $\mathbb{P}^r$-bundle structure $\bar{\psi} : Z = \mathbb{P}_S(F) \to S$ for some rank $r+1$ vector bundle F over a smooth base S,

(ii) $N_{Z/X}|_{Z_s} \cong \mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)}$ for each $\bar{\psi}$-fiber Z_s, $s \in S$.

To construct the corresponding flops, we blow up X along Z to get $\phi : Y \to X$ and the exceptional divisor E is a $\mathbb{P}^r \times \mathbb{P}^r$-bundle over S. Then we can blow down E along another fiber direction to get $\phi' : Y \to X'$, with exceptional loci $\bar{\psi}' : Z' = \mathbb{P}_S(F') \to S$ for F' being another rank $r+1$ vector bundle over S and also $N_{Z'/X'}|_{\psi'-\text{fiber}} \cong \mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)}$.

We call $f : X \dashrightarrow X'$ constructed as above an *ordinary $\mathbb{P}^r$ flop*. The various sets and maps are summarized in the following commutative diagram.

$$
\begin{tikzcd}
 & E = Z \times_S Z' \arrow[dl, "\bar\phi"'] \arrow[dr, "\bar\phi'"] \arrow[r, hook, "j"] & Y \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\
Z = \mathbb{P}_S(F) \arrow[r, hook, "i"] \arrow[dr, "\bar\psi"'] & X \arrow[dr, "\psi"] & Z' = \mathbb{P}_S(F') \arrow[r, hook, "i'"] \arrow[dl, "\bar\psi'"'] & X' \arrow[dl, "\psi'"] \\
 & S \arrow[r, hook, "j'"'] & \overline{X} &
\end{tikzcd}
$$

When S consists of a point, we call f a *simple* $\mathbb{P}^r$ *flop*.

2.2. Equivalence of Chow motives

Instead of comparing special cohomology groups, we are devoted to the universal cohomology theory, namely Grothendieck's category of Chow motives. General references of Chow motives can be found in [16].

Let $\mathcal{M}$ be the category of motives (over $\mathbb{C}$). For each smooth variety X, one associates an object $\hat{X}$ in $\mathcal{M}$. A morphism from $\hat{X}_1$ to $\hat{X}_2$ is a correspondence $U \in A^*(X_1 \times X_2)$ which has induced maps on T-valued points $\mathrm{Hom}(\hat{T}, \hat{X}_i)$:

$$U_T : A^*(T \times X_1) \xrightarrow{U\circ} A^*(T \times X_2).$$

and the composition law is given by

$$V \circ U = p_{13*}(p_{12}^* U . p_{23}^* V)$$

where $U \in A^*(X_1 \times X_2)$, $V \in A^*(X_2 \times X_3)$ and $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ are the projection maps.

The basic tool in motives is *Manin's identity principle*: Let U, $V \in \mathrm{Hom}(\hat{X}, \hat{X}')$. Then $U = V$ if and only if $U_T = V_T$ for all T.

For a $\mathbb{P}^r$ flop $f : X \dashrightarrow X'$, to see that the graph closure $[\bar{\Gamma}_f] \in A^*(X \times X')$ identifies the Chow motives $\hat{X}$ of X and $\hat{X}'$ of X', our strategy is to apply the identity principle to show that $\mathcal{F}^* \circ \mathcal{F} = \Delta_X$ and $\mathcal{F} \circ \mathcal{F}^* = \Delta_{X'}$.

For any T, $\mathrm{id}_T \times f : T \times X \dashrightarrow T \times X'$ is also an ordinary $\mathbb{P}^r$ flop. Hence to prove that $\mathcal{F}^* \circ \mathcal{F} = \Delta_X$, we only need to show that $\mathcal{F}^*\mathcal{F} = \mathrm{id}$ on $A^*(X)$ for any ordinary $\mathbb{P}^r$ flop. The associated map on Chow groups of the correspondence $\mathcal{F}$ is

$$\mathcal{F} : A^*(X) \to A^*(X'); \quad W \mapsto p'_*(\bar{\Gamma}_f . p^* W) = \phi'_* \phi^* W.$$

Let $\tilde{W}$ be the proper transform of W in Y and W' be the proper transform of W in X'. By the precise formulae for pull-back from the intersection theory ([4], Theorem 6.7) and dimensional consideration, we get

$\phi^* W = \tilde{W}$ and thus $\mathcal{F}W = W'$. By more delicate dimensional computation, we find that the error term of $\phi^* W$ and $\phi'^* W'$ contains both fibers of ϕ as well as ϕ' and thus $\mathcal{F}^*\mathcal{F}W = W$. By symmetry, $\mathcal{F}\mathcal{F}^* W' = W'$. Hence we have our first result:

Theorem 4 *For an ordinary $\mathbb{P}^r$ flop $f : X \dashrightarrow X'$, the graph closure $\mathcal{F} := [\bar{\Gamma}_f]$ induces $\hat{X} \cong \hat{X}'$ via $\mathcal{F}^* \circ \mathcal{F} = \Delta_X$ and $\mathcal{F} \circ \mathcal{F}^* = \Delta_{X'}$.*

Using again the fact that the difference of $\phi^*\alpha_i$ and $\phi'^*\mathcal{F}\alpha_i$ has positive fiber dimension in both the ϕ direction and the ϕ' direction, it follows that $\mathcal{F}$ preserves the Poincaré pairing.

Corollary 5 *Let $f : X \dashrightarrow X'$ be a $\mathbb{P}^r$ flop. If $\dim \alpha_1 + \dim \alpha_2 = \dim X$, then*

$$(\mathcal{F}\alpha_1.\mathcal{F}\alpha_2) = (\alpha_1.\alpha_2).$$

Remark 6 *(i) Since every geometric cohomology theory (a graded ring functor H^* with Poincaré duality, Künneth formula and a cycle map $A^* \to H^*$ etc.) factors through $\mathcal{M}$, the theorem also holds on such a specialized theory.*

(ii) However the ring structure, i.e. the cohomology product structure, is not preserved under the correspondence $\mathcal{F}$ (c.f. next section). To investigate a general product structure $*$ on $H^*(X)$, let $\{T_i\}$ be a cohomology basis and $\{T^i\}$ be the dual basis with $(T^i.T_j) = \delta_{ij}$. Write

$$T_i * T_j = \sum\nolimits_k c_{ijk} T^k.$$

We usually require that $\int_X T_i * T_j = (T_i.T_j)$. Then the structure constants

$$c_{ijk} = (T_i * T_j.T_k) = (T_i * T_j * T_k).$$

Hence under the Poincaré pairing, $*$ is determined by its triple product.

2.3. The defect of triple products

In this section, I am going to determine the defect of triple products under a simple ordinary flop.

Let $f : X \dashrightarrow X'$ be a simple $\mathbb{P}^r$ flop. Let h be the hyperplane class of $Z = \mathbb{P}^r$ and h' be the hyperplane class of Z'. Let also $x = \bar{\phi}^* h = [h \times \mathbb{P}^r]$, $y = \bar{\phi}'^* h' = [\mathbb{P}^r \times h']$ in $E = \mathbb{P}^r \times \mathbb{P}^r$. First of all, we seek out the correspondence of classes in Z and Z':

Lemma 7 *For classes inside Z, we have*

$$\mathcal{F}[h^k] = (-1)^{r-k}[h'^k].$$

In particular, $\mathcal{F}[C] = -[C']$ with C, C' being the line classes in Z, Z' respectively.

Next we determine the difference of two pull-backs of α and $\mathcal{F}\alpha$ with classes α in X. The proof given below is slightly more concise than the original one in [8] from the structural viewpoint.

Lemma 8 *For a class $\alpha \in H^{2k}(X)$ with $k \le r$, let $\alpha' = \mathcal{F}\alpha$ in X'. Then*

$$\phi'^*\alpha' = \phi^*\alpha + (\alpha.h^{r-k})\, j_*\frac{x^k - (-y)^k}{x+y}.$$

Proof. Since the difference $\phi'^*\alpha' - \phi^*\alpha$ has support in E, we may write $\phi'^*\alpha' - \phi^*\alpha = j_*\lambda$ for some $\lambda \in H^{2(k-1)}(E)$. Then

$$(\phi'^*\alpha')|_E - (\phi^*\alpha)|_E = j^*j_*\lambda = c_1(N_{E/Y})\lambda = -(x+y)\lambda.$$

By the Lefschetz hyperplane theorem, we have

$$\begin{aligned}\lambda &= -\frac{1}{x+y}((\phi'^*\alpha')|_E - (\phi^*\alpha)|_E) = -\frac{1}{x+y}(\bar{\phi}'^*(\alpha'|_{Z'}) - \bar{\phi}^*(\alpha|_Z))\\ &= -\frac{1}{x+y}\big(\bar{\phi}'^*((\alpha'.h'^{r-k})h'^k)) - \bar{\phi}^*((\alpha.h^{r-k})h^k)\big).\end{aligned}$$

Since $\mathcal{F}$ preserves the Poincaré pairing,

$$(\alpha'.h'^{r-k}) = (\mathcal{F}\alpha.\mathcal{F}((-1)^k h^{r-k})) = (-1)^k(\alpha.h^{r-k}).$$

Hence we have

$$\lambda = -(\alpha.h^{r-k})\frac{\bar{\phi}'^*(-1)^k h'^k - \bar{\phi}^* h^k}{x+y} = (\alpha.h^{r-k})\frac{x^k - (-y)^k}{x+y}.$$

□

These formulae allow us to compare the triple products of classes in X and X'. Besides I would like to simplify the proof in [8] a bit.

Proposition 9 *For a simple $\mathbb{P}^r$-flop $f : X \dashrightarrow X'$, let $\alpha_i \in H^{2k_i}(X)$, with $k_i \le r$, $k_1 + k_2 + k_3 = \dim X = 2r+1$. Then*

$$(\mathcal{F}\alpha_1.\mathcal{F}\alpha_2.\mathcal{F}\alpha_3) = (\alpha_1.\alpha_2.\alpha_3) + (-1)^r(\alpha_1.h^{r-k_1})(\alpha_2.h^{r-k_2})(\alpha_3.h^{r-k_3}).$$

Proof. Since for all $i = 1,2,3$, $\phi'^*\mathcal{F}\alpha_i = \phi^*\alpha_i + j_*\lambda_i$ with

$$\lambda_i = (\alpha_i.h^{r-k_i})\, j_*\frac{x^{k_i} - (-y)^{k_i}}{x+y}$$

which contains both fiber directions of $\bar{\phi}$ and $\bar{\phi}'$, we have

$$\begin{aligned}(\mathcal{F}\alpha_1.\mathcal{F}\alpha_2.\mathcal{F}\alpha_3) &= (\phi'^*\mathcal{F}\alpha_1.\phi'^*\mathcal{F}\alpha_2.(\phi^*\alpha_3 + j_*\lambda_3)) = (\phi'^*\mathcal{F}\alpha_1.\phi'^*\mathcal{F}\alpha_2.\phi^*\alpha_3)\\ &= ((\phi^*\alpha_1 + j_*\lambda_1).(\phi^*\alpha_2 + j_*\lambda_2).\phi^*\alpha_3).\end{aligned}$$

Among the resulting terms, the first term is clearly equal to $(\alpha_1.\alpha_2.\alpha_3)$. For those terms with two pull-backs like $\phi^*\alpha_1.\phi^*\alpha_3$, the intersection values are zero since the remaining part contains the ϕ fiber. The remaining term contributes

$$\begin{aligned}&\phi^*\alpha_3.j_*\frac{x^{k_1}-(-y)^{k_1}}{x+y}.j_*\frac{x^{k_2}-(-y)^{k_2}}{x+y}\\&=-\phi^*\alpha_3.j_*\big((x^{k_1}-(-y)^{k_1})(x^{k_2-1}+x^{k_2-2}(-y)+\cdots+(-y)^{k_2-1})\big)\end{aligned}$$

times $(\alpha_1.h^{r-k_1})(\alpha_2.h^{r-k_2})$. The terms with non-trivial contribution must contain y^r, hence there is only one such term, namely (notice that $k_1+k_2+k_3=2r+1$)

$$-(-y)^{k_1}\times x^{k_2-1-(r-k_1)}(-y)^{r-k_1}=-(-1)^r x^{r-k_3}y^r$$

and the contribution is $(-1)^r(\alpha_1.h^{r-k_1})(\alpha_2.h^{r-k_2})(\alpha_3.h^{r-k_3})$. $\square$

3. Quantum corrections

The theorem above on triple product suggests that one needs to correct the product structure by some contributions from the extremal ray. In this section we illustrate the reason why the quantum corrections attached to the extremal ray exactly remedy the defect of the ordinary product for simple ordinary flops.

3.1. Gromov-Witten invariants

We use [3] as our general reference on moduli spaces of stable maps, Gromov-Witten theory and quantum cohomology.

Let $\beta\in NE(X)$, the Mori cone of numerical classes of effective one cycles. Let $\overline{M}_{g,n}(X,\beta)$ be the moduli space of n-pointed stable maps $f:(C;x_1,\ldots,x_n)\to X$ from a nodal curve C with arithmetic genus $g(C)=g$ and with degree $[f(C)]=\beta$. Let $e_i:\overline{M}_{g,n}(X,\beta)\to X$ be the evaluation morphism $f\mapsto f(x_i)$. The Gromov-Witten invariant for classes $\alpha_i\in H^*(X)$, $1\le i\le n$, is given by

$$\langle\alpha_1,\ldots,\alpha_n\rangle_{g,n,\beta}:=\int_{[\bar{M}_{g,n}(X,\beta)]^{virt}}e_1^*\alpha_1\cdots e_n^*\alpha_n.$$

The idea of Gromov-Witten invariants is that if we want to compute the relation of classes in X via stable maps, then we may use these evaluation morphisms to pull back the classes to the moduli space $\bar{M}_{g,n}(X,\beta)$ and take integration. There are something subtle here. Because the moduli space usually does not have correct dimension, Li and Tian constructed the virtual moduli cycle $[\bar{M}_{g,n}(X,\beta)]^{virt}$ to have the expected dimension [12]. The virtual (expected) dimension of $\bar{M}_{g,n}(X,\beta)$ is given by

$$(c_1(X).\beta) + \dim X(1-g) + (3g-3) + n.$$

In our case, $\psi : X \to \bar{X}$ is a simple $\mathbb{P}^r$ flopping contraction with $Z = \mathbb{P}^r \subset X$ and $N_{Z/X} \cong \mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)}$. If we deal with the case of $\beta = d\ell$ with $\ell = [C]$, the extremal ray contracted by ψ, then since $(K_X.\ell) = 0$, for $g = 0$, the virtual dimension of $\bar{M}_{0,n}(X, d\ell)$ equals $2r + 1 + (n-3)$.

In practice, we may represent $[\bar{M}_{0,n}(X, d\ell)]^{virt}$ by the Euler class of the obstruction bundle

$$U_d = R^1\rho_* e_{n+1}^* N_{Z/X},$$

where $\rho : \bar{M}_{0,n+1}(\mathbb{P}^r, d) \to \bar{M}_{0,n}(\mathbb{P}^r, d)$ is the forgetting morphism. Then

$$\int_{[\bar{M}_{0,n}(X,d\ell)]^{virt}} e_1^*\alpha_1 \cdots e_n^*\alpha_n = \int_{\bar{M}_{0,n}(\mathbb{P}^r,d)} e_1^*(\alpha_1|_{\mathbb{P}^r}) \cdots e_n^*(\alpha_n|_{\mathbb{P}^r}).e(U_d).$$

3.2. Quantum product

Let $T = \sum t_i T_i$ with $\{T_i\}$ a cohomology basis and t_i being formal variables. Let $\{T^i\}$ be the dual basis with $(T^i.T_j) = \delta_{ij}$. The (genus zero) pre-potential combines all n-point functions together:

$$\Phi(T) = \sum\nolimits_{n=0}^{\infty} \sum\nolimits_{\beta \in NE(X)} \frac{1}{n!} \langle T^n \rangle_\beta \, q^\beta,$$

where $\langle T^n \rangle_\beta = \langle T, \ldots, T \rangle_{0,n,\beta}$. The *big quantum product* is defined by

$$T_i * T_j = \sum\nolimits_k \Phi_{ijk} T^k$$

where

$$\Phi_{ijk} = \frac{\partial^3\Phi}{\partial t_i \partial t_j \partial t_k} = \sum\nolimits_{n=0}^{\infty} \sum\nolimits_{\beta \in NE(X)} \frac{1}{n!} \langle T_i, T_j, T_k, T^n \rangle_\beta \, q^\beta.$$

The $n = 0$ part $\Phi_{ijk}(0)$ gives the *small quantum product*, that is,

$$T_i * T_j = \sum\nolimits_k \sum\nolimits_{\beta \in NE(X)} \langle T_i, T_j, T_k \rangle_\beta \, q^\beta T^k.$$

Let $f : X \dashrightarrow X'$ be a simple $\mathbb{P}^r$ flop. Since X and X' have the same Poincaré pairing under $\mathcal{F}$, in order to compare their quantum products we only need to compare their n-point functions. For three-point functions, write

$$\begin{aligned}\langle \alpha_1, \alpha_2, \alpha_3 \rangle &:= \sum\nolimits_{\beta \in NE(X)} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,\beta} \, q^\beta \\ &= (\alpha_1.\alpha_2.\alpha_3) + \sum\nolimits_{d \in \mathbb{N}} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{d\ell} \, q^{d\ell} + \sum\nolimits_{\beta \notin \mathbb{Z}\ell} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_\beta \, q^\beta.\end{aligned}$$

The difference $(\mathcal{F}\alpha_1.\mathcal{F}\alpha_2.\mathcal{F}\alpha_3)-(\alpha_1.\alpha_2.\alpha_3)$ has already been determined. The next step is to compute the middle term, namely quantum corrections coming from the extremal ray $\ell = [C]$. We will see that the three-point functions attached to the extremal ray exactly remedy the defect caused by the classical product. In the end, we will achieve that the remaining terms are invariant under a simple flop in the sense of analytic continuation over the extended Kähler moduli space.

3.3. GW invariants attached to extremal rays

We derive a precise formula for this case.

Theorem 10 *For all $\alpha_i \in H^{2l_i}(X)$ with $1 \le l_i \le r$ and $\sum_{i=1}^n l_i = 2r+1+(n-3)$, there are recursively determined universal constants $N_{l_1,\ldots,l_n}$, which are independent of d, such that for $n \le 3$, $N_* \equiv 1$ and*

$$\langle \alpha_1,\ldots,\alpha_n\rangle_{0,n,d} = (-1)^{(d-1)(r+1)} N_{l_1,\ldots,l_n} d^{n-3}(\alpha_1.h^{r-l_1})\cdots(\alpha_n.h^{r-l_n}).$$

Equivalently,

$$\int_{\bar{M}_{0,n}(\mathbb{P}^r,d)} e_1^*h^{l_1}\cdots e_n^*h^{l_n}.e(U_d) = (-1)^{(d-1)(r+1)} N_{l_1,\ldots,l_n} d^{n-3}.$$

The proof consists of two main steps. The first step is to use the theory on Euler data [13] to compute certain twisted Gromov-Witten invariants of concave bundle spaces. Originally it was used to compute invariants without marked points (i.e. no cohomology insertions), but it works only for *critical bundles* and does not apply to our case. Yet, through a closer study, the theory of Euler data does lead to the determination of one-point invariants with descendents (i.e. ψ classes).

The second step is to use the divisor relation on the genus zero stable map moduli spaces [9] to reduce the multiple marked points invariants to the ones with fewer marked points. I will now sketch both steps.

For step 1, we need two moduli spaces other than the original stable map moduli:

$$\begin{tikzcd} M_d = \bar{M}_{0,0}(\mathbb{P}^1\times\mathbb{P}^r,(1,d)) \arrow[r, "\pi"] \arrow[d, "\varphi"] & \bar{M}_{0,0}(\mathbb{P}^r,d) \\ N_d \cong \mathbb{P}^{(r+1)(d+1)-1} & \end{tikzcd}$$

Here M_d is the graph space and N_d is the linear sigma model. A point in N_d is denoted by $(z_{is})_{i=0,\ldots r;\ s=0,\ldots,d}$, which corresponds to the map

$$(w_0:w_1)\mapsto (\sum z_{0s}w_0^s w_1^{d-s}:\cdots:\sum z_{rs}w_0^s w_1^{d-s}).$$

The torus $T = (\mathbb{C}^\times)^{r+1}$ acts on $\mathbb{P}^r$ with weight $\lambda_0, \ldots, \lambda_r$ and $\mathbb{C}^\times$ acts on $\mathbb{P}^1$ by $t(w_0, w_1) = (tw_0, w_1)$ with weight $\alpha, 0$. The equivariant cohomology of N_d is given by

$$H_G^*(N_d) = \mathbb{Q}[\alpha, \lambda_0, \ldots, \lambda_r][\kappa] / \prod\nolimits_{i,s} (\kappa - (\lambda_i + s\alpha)),$$

where κ is the equivariant hyperplane class. Let

$$Q_d = \varphi_* \pi^* e_T(U_d) \in H_G(N_d),$$

Then Lian, Liu and Yau [13] show that

$$Q_d = (-1)^{(d-1)(r+1)} \prod\nolimits_{m=1}^{d-1} (\kappa - m\alpha)^{r+1}.$$

For $I_d : \mathbb{P}^r = N_0 \to N_d$, $(a_0, \cdots, a_r) \mapsto (a_0 w_1^d, \cdots, a_r w_1^d)$, the Atiyah-Bott localization theorem implies that

$$I_d^* Q_d = \prod\nolimits_{j=0}^{r} \prod\nolimits_{m=1}^{d} (h - \lambda_j - m\alpha). e_{1*} \left(\frac{e_T(U_d)}{\alpha(\alpha + \psi_1)} \right),$$

where $\psi_1 = c_1^G(L_1)$ and L_i is the i-th cotangent line bundle. Recall

$$\left\langle \tau_{k_1}(h^{l_1}), \cdots, \tau_{k_n}(h^{l_n}) \right\rangle_d := \int_{\bar{M}_{0,n}(\mathbb{P}^r, d)} \left(\prod\nolimits_{i=1}^{n} \psi_i^{k_i} e_i^* h^{l_i} \right) . e_T(U_d).$$

Then we combine the above two formulas of Q_d to obtain:

Theorem 11 (One point invariants with ψ class) *For $l + k = 2r - 1$,*

$$\left\langle \tau_k(h^l) \right\rangle_d = (-1)^{(d-1)(r+1)} \frac{(-1)^{k-(r+1)}}{d^{k+2}} C_r^{k+1}.$$

Now we review step 2. Recall the divisor relation of Lee and Pandharipande [9]: For $L \in \mathrm{Pic}(X)$ and $i \neq j$,

$$\begin{aligned} & e_i^* L \cap [\bar{M}_{0,n}(X, \beta)]^{virt} \\ & = (e_j^* L + (\beta.L)\psi_j) \cap [\bar{M}_{0,n}(X, \beta)]^{virt} - \sum_{\beta_1 + \beta_2 = \beta} (\beta_1.L)[D_{i,\beta_1|j,\beta_2}]^{virt} \end{aligned}$$

in $A^*(\bar{M}_{0,n}(X, \beta))$. That is, we may switch marked points if we can handle boundary divisors and ψ classes. In fact we can do that and obtain:

Theorem 12 (Two point invariants) *The only non-trivial two point invariant (without ψ classes) is given by*

$$\langle h^r, h^r \rangle_d = (-1)^{(d-1)(r+1)} \frac{1}{d}.$$

Recall for the equivalent form we need to show: For all $d \in \mathbb{N}$, $\sum_{i=1}^n l_i = 2r + 1 + (n-3)$,

$$\langle h^{l_1}, \ldots, h^{l_n} \rangle_d = (-1)^{(d-1)(r+1)} N_{l_1,\ldots,l_n} d^{n-3}.$$

For $n \geq 3$ and for any 3 markings i, j and k, $\psi_j = [D_{ik|j}]^{virt}$, the divisor relation can be re-written as

$$e_i^* L = e_j^* L + \sum_{\beta_1+\beta_2=\beta} ((\beta_2.L)[D_{ik,\beta_1|j,\beta_2}]^{virt} - (\beta_1.L)[D_{i,\beta_1|jk,\beta_2}]^{virt}).$$

This leads to the

Theorem 13 (Final Reduction) *The following reduction formula holds for $n \geq 3$:*

$$\begin{aligned} &\langle h^{l_1+1}, h^{l_2}, h^{l_3}, \ldots \rangle_{n,d} \\ &\quad = \langle h^{l_1}, h^{l_2+1}, h^{l_3}, \ldots \rangle_{n,d} \\ &\qquad + d\langle h^{l_1+l_3}, h^{l_2}, \ldots \rangle_{n-1,d} - d\langle h^{l_1}, h^{l_2+l_3}, \ldots \rangle_{n-1,d}. \end{aligned}$$

The desired formula then follows by induction.

3.4. Analytic continuations along extremal rays

Theorem 3.1, together with some algebraic manipulations, implies that the quantum corrections attached to the extremal ray exactly remedy the defect caused by the classical product and the big quantum products restricted to exceptional curve classes are invariant under simple ordinary flops. There are Novikov variables q^β involved in these transformations:

$$\mathcal{F}(q^\beta) = q^{\mathcal{F}\beta}.$$

To put the result into perspective, we interpret the change of variables in terms of analytic continuation over the extended complexified Kähler moduli space.

The quantum cohomology is parameterized by the complexified Kähler class $\omega = B + iH$ with $q^\beta = \exp(2\pi i(\omega.\beta))$, where $B \in H^{1,1}_{\mathbb{R}}(X)$ and $H \in \mathcal{K}_X$, the Kähler cone of X. For a simple $\mathbb{P}^r$ flop $X \dashrightarrow X'$, $\mathcal{F}$ identifies $H^{1,1}$, A_1 and the Poincaré pairing $(-,-)$ on X and X'. Then by applying Theorem 3.1, $\langle \alpha_1, \alpha_2, \alpha_3 \rangle^X$ restricted to $\mathbb{Z}\ell$ converges in the region

$$H^{1,1}_+ = \{\omega \,|\, (H.\ell) > 0\} \supset H^{1,1}_{\mathbb{R}} \times i\,\mathcal{K}_X$$

and the corresponding geometric series equals

$$(\alpha_1.\alpha_2.\alpha_3) + (\alpha_1.h^{r-l_1})(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3}) \frac{e^{2\pi i(\omega.\ell)}}{1 + (-1)^r e^{2\pi i(\omega.\ell)}}.$$

This is a well-defined analytic function of ω on the whole $H^{1,1}$, which defines the analytic continuation of $\langle\alpha_1,\alpha_2,\alpha_3\rangle^X$ from $H^{1,1}_{\mathbb{R}} \times i\,\mathcal{K}_X$ to $H^{1,1}$.

Similarly, $\langle\mathcal{F}\alpha_1,\mathcal{F}\alpha_2,\mathcal{F}\alpha_3\rangle^{X'}$ restricted to $\mathbb{Z}\ell'$ converges in

$$\{\omega\,|\,(H.\ell')>0\} = \{\omega\,|\,(H.\ell)<0\} = H^{1,1}_{-} \supset H^{1,1}_{\mathbb{R}} \times i\,\mathcal{K}_{X'}.$$

After the change of variable replacing ℓ' by $-\ell$ and the identification of $(\mathcal{F}\alpha_i.h'^{(r-l_i)})$ with $(-1)^{l_i}(\alpha_i.h^{r-l_i})$, it equals

$$(\mathcal{F}\alpha_1.\mathcal{F}\alpha_2.\mathcal{F}\alpha_3) - (\alpha_1.h^{r-l_1})(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3})\frac{e^{-2\pi i(\omega.\ell)}}{1+(-1)^r e^{-2\pi i(\omega.\ell)}}$$

which is the analytic continuation of the previous one from $H^{1,1}_{+}$ to $H^{1,1}_{-}$.

This illustrates that *the three-point functions attached to the extremal ray exactly remedy the defect caused by the classical product.*

For invariance of big quantum product restricted to exceptional curve classes, we need to compare $n = 3+k$ point invariants with $k \ge 1$. By Theorem 3.1 again, we get

$$\langle\alpha_1,\ldots,\alpha_n\rangle = N_{l_1,\ldots,l_n}(\alpha_1.h^{r-l_1})\cdots(\alpha_n.h^{r-l_n})\left(q^\ell\frac{d}{dq^\ell}\right)^k\frac{(-1)^{r+1}}{1-(-1)^{r+1}q^\ell}.$$

Similarly, since $(-1)^{\sum l_i} = (-1)^{k+1}$, $\langle\mathcal{F}\alpha_1,\ldots,\mathcal{F}\alpha_n\rangle$ equals

$$(-1)^{k+1}N_{l_1,\ldots,l_n}(\alpha_1.h^{r-l_1})\cdots(\alpha_n.h^{r-l_n})\left(q^{\ell'}\frac{d}{dq^{\ell'}}\right)^k\frac{(-1)^{r+1}}{1-(-1)^{r+1}q^{\ell'}}.$$

Taking into account of

$$q^{-\ell}\frac{d}{dq^{-\ell}} = -q^\ell\frac{d}{dq^\ell} \quad\text{and}\quad \frac{1}{1-(-1)^{r+1}q^{-\ell}} = 1-\frac{1}{1-(-1)^{r+1}q^\ell}$$

we get $\langle\mathcal{F}\alpha_1,\ldots,\mathcal{F}\alpha_n\rangle = \langle\alpha_1,\ldots,\alpha_n\rangle$ for all $k \ge 1$ ($n \ge 4$).

It is conjectured that the total series Φ^X_{ijk}, converges for $B \in \mathcal{K}_X$, at least for B large enough, hence the large radius limit goes back to the classical cubic product. The Novikov variables $\{q^\beta\}_{\beta\in NE(X)}$ are introduced to avoid the convergence issue.

Since $\mathcal{K}_X \cap \mathcal{K}_{X'} = \emptyset$ for non-isomorphic K-equivalent models, the collection of Kähler cones among them form a chamber structure. The conjectural canonical isomorphism

$$\mathcal{F} : H^*(X) \cong H^*(X')$$

assigns to each model X a *coordinate system* $H^*(X)$ of the fixed H^* and $\mathcal{F}$ serves as the *(linear) transition function.* The conjecture asserts that $\Phi^{X'}_{ijk}$ can be *analytically continued* from $\mathcal{K}_{X'}$ to $\mathcal{K}_X$ and agrees with Φ^X_{ijk}. Equivalently, Φ_{ijk} is well-defined on $\mathcal{K}_X \cup \mathcal{K}_{X'}$ which verifies the *functional equation*

$$\mathcal{F}\Phi_{ijk}(\omega, T) \cong \Phi_{ijk}(\omega, \mathcal{F}T).$$

4. Degeneration analysis

To achieve the invariance of big quantum product, non-extremal curve classes need to be analyzed.

4.1. Cohomology reduction to local models

The main purpose of this section is to reduce cohomology classes in general X to cohomology classes in local models.

Given a $\mathbb{P}^r$ flop $f : X \dashrightarrow X'$, the *deformations to the normal cone* on X is the blowing-up $\Phi : W \to X \times \mathbb{A}^1$ along $Z \times \{0\}$. $W_t \cong X$ for all $t \neq 0$ and $W_0 = Y_1 \cup Y_2$ with $j_i : Y_i \hookrightarrow W_0$ the inclusion maps for $i = 1, 2$. Here $Y_1 = Y$ with $\phi = \Phi|_Y : Y \to X$ is the blowing-up along Z and $Y_2 = \tilde{E} = \mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O})$ where $p = \Phi|_{\tilde{E}} : \tilde{E} \to Z \subset X$ is the compactified normal bundle. $Y \cap \tilde{E} = E = \mathbb{P}_Z(N_{Z/X})$ is the ϕ-exceptional divisor which consists of the infinity part of $\tilde{E}$. Similarly we have $\Phi' : W' \to X' \times \mathbb{A}^1$ and $W'_0 = Y' \cup \tilde{E}'$. By definition of ordinary flops, $Y = Y'$ and $E = E'$. In fact $\tilde{E} \cong \tilde{E}'$ too, but they are glued into Y in a different manner (up to a twist), thus $W_0 \ncong W'_0$.

Since the family $W \to \mathbb{A}^1$ comes from a trivial family, all cohomology classes $\alpha \in H^*(X, \mathbb{Z})^{\oplus n}$ have global liftings and the restriction $\alpha(t)$ on W_t is defined for all t. The class $\alpha(0)$ can be represented by $(j_1^*\alpha(0), j_2^*\alpha(0)) = (\alpha_1, \alpha_2)$ with $\alpha_i \in A^*(Y_i)$ such that

$$\iota_1^*\alpha_1 = \iota_2^*\alpha_2 \quad \text{and} \quad \phi_*\alpha_1 + p_*\alpha_2 = \alpha.$$

Such representatives are not unique. The flexibility on different choices is of key importance. Actually for e being a class in E, if $\alpha(0) = (\alpha_1, \alpha_2)$ then it can also be represented by

$$\alpha(0) = (\alpha_1 - \iota_{1*}e, \alpha_2 + \iota_{2*}e).$$

We start with the representative $(\phi^*\alpha, p^*(\alpha|_Z))$ for $\alpha(0)$ and the representative $(\phi'^*\mathcal{F}\alpha, p'^*(\mathcal{F}\alpha|_{Z'}))$ for $\mathcal{F}\alpha(0)$. Then we can modify the choices $\phi^*\alpha$ and $\phi'^*\mathcal{F}\alpha$ by adding suitable classes in E to make them equal. This is possible since

$$\phi^*\alpha - \phi'^*\mathcal{F}\alpha \in \iota_{1*}H^*(E).$$

Finally, we can show that for representatives $\alpha(0) = (\alpha_1, \alpha_2)$ and $\mathcal{F}\alpha(0) = (\alpha'_1, \alpha'_2)$,

$$\text{if} \quad \alpha_1 = \alpha'_1 \quad \text{then} \quad \mathcal{F}\alpha_2 = \alpha'_2.$$

Here we must mention that the ordinary flop f induces an ordinary flop

$$\tilde{f} : \tilde{E} \dashrightarrow \tilde{E}'$$

on the local model, so the graph closure $\mathcal{F}$ of $\tilde{f}$ also gives a correspondence of $H^*(\tilde{E})$ and $H^*(\tilde{E}')$.

4.2. Degeneration formula

The *degeneration formula* expresses the absolute invariants of X in terms of the relative invariants of the two smooth pairs (Y_1, E) and (Y_2, E) stated in §4.1.

We start with the formulation given by J. Li [11]. It reads:

$$\langle \alpha \rangle^X_{g,k,\beta} = \sum\nolimits_\eta C_\eta \left[\langle j_1^* \alpha(0) \rangle^{(Y_1,E)}_{\Gamma_1} \cdot \langle j_2^* \alpha(0) \rangle^{(Y_2,E)}_{\Gamma_2} \right]_0 .$$

Here for given genus g, number of marked points k and $\beta \in NE(X)$, $\eta = (\Gamma_1, \Gamma_2, I)$ with $I = (I_L, I_R)$ runs through all equivalence classes of *admissible triples*. Namely, each Γ_i is an *admissible graph* without edges which consists of vertexes V_{Γ_i} (connected components), legs L_{Γ_i} (marked points), roots R_{Γ_i} (gluing points), attaching map $L_{\Gamma_i} \coprod R_{\Gamma_i} \to V_{\Gamma_i}$, genus function $g_i : V_{\Gamma_i} \to \mathbb{N} \cup \{0\}$, degree function $\beta_i : V_{\Gamma_i} \to NE(Y_i)$, ordering of marked points $I_i : L_{\Gamma_i} \to \{1, \cdots, k_i := |L_{\Gamma_i}|\}$ and contact order $\mu_i : R_{\Gamma_i} \to \mathbb{N}$.

The graph Γ_i is pre-connected in the sense that either $v_i := |V_{\Gamma_i}| = 1$ or $R_{\Gamma_i} \to V_{\Gamma_i}$ is surjective. The two admissible graphs Γ_1 and Γ_2 glues together along roots via $I_R : R_{\Gamma_1} \cong R_{\Gamma_2}$ with $\mu_1 = \mu_2$ under the identification. Two vertexes in the new graph Γ is assigned an edge connecting them whenever they are related via roots. These data satisfy ceratin compatibility identities. Namely $k_1 + k_2 = k$, the total ordering $I_L : L_{\Gamma_1} \coprod L_{\Gamma_2} \to \{1, \cdots, k\}$ preserves the ordering of I_1 and I_2 and $g - 1 = \sum_{v \in V_{\Gamma_1}} (g_1(v) - 1) + \sum_{v \in V_{\Gamma_2}} (g_2(v) - 1) + \rho$, where $\rho = |R_{\Gamma_i}|$ is the number of roots.

The crucial condition is that Γ is connected. In particular, $\rho = 0$ if and only if that one of the Γ_i is empty. Also the total degree $\beta_i := \sum_{v \in V_{\Gamma_i}} \beta_i(v) \in NE(Y_i)$ satisfies the splitting relation

$$\phi_* \beta_1 + p_* \beta_2 = \beta.$$

The constants $C_\eta = m(\mu)/|\mathrm{Aut}\,\eta|$, where $m(\mu) = \prod \mu_i$ and $\mathrm{Aut}\,\eta = \{\, \sigma \in S_\rho \mid \eta^\sigma = \eta \,\}$. For each η there is a *gluing morphism* for moduli spaces of relative stable maps under prescribed constraints:

$$\Phi_\eta : \bar{M}_{\Gamma_1}(Y_1, E) \times_{E^\rho} \bar{M}_{\Gamma_2}(Y_2, E) \to \bar{M}_{g,k,\beta}(W/\mathbb{A}^1),$$

which is finite étale of degree $|\mathrm{Aut}\,\eta|$ onto its image $\bar{M}_\eta(W_0)$.

Each moduli space has perfect obstruction theory. The *virtual moduli cycle* of $\bar{M}_{g,k,\beta}(W/\mathbb{A}^1)$ is flat over $\mathbb{A}^1$ and its zero fiber is made up by fiber products of those virtual moduli cycles of $\bar{M}_{\Gamma_1}(Y_1, E)$ and $\bar{M}_{\Gamma_2}(Y_2, E)$ via Φ_η. The relative invariants ($i = 1, 2$) are

$$\langle j_i^*\alpha(0)\rangle_{\Gamma_i}^{(Y_i,E)} \equiv q_{i*}\left(ev_i^* j_i^*\alpha(0) \cap [M_{\Gamma_i}(Y_i, E)]^{virt}\right) \in H_*(E^\rho, \mathbb{Q}),$$

where $ev_i : M_{\Gamma_i}(Y_i, E) \to Y_i^{k_i}$ and $q_i : M_{\Gamma_i}(Y_i, E) \to E^\rho$ are evaluation maps on marked points and gluing points respectively. This formulation will be used in dealing with examples at the end of this section.

On the other hand, the following (equivalent) numerical form originally obtained by A. Li and Y. Ruan [10] will be used in our proof:

$$\langle\alpha\rangle_{g,n,\beta}^X = \sum_I \sum_{\eta\in\Omega_\beta} C_\eta \left\langle \alpha_1 \,\middle|\, e_I, \mu \right\rangle_{\Gamma_1}^{\bullet(Y_1,E)} \left\langle \alpha_2 \,\middle|\, e^I, \mu \right\rangle_{\Gamma_2}^{\bullet(Y_2,E)}$$

where $\{e_i\}$ is a basis of $H^*(E)$ with $\{e^i\}$ its dual basis and $\{e_I\}$ forms a basis of $H^*(E^\rho)$ with dual basis $\{e^I\}$ where $|I| = \rho$, $e_I = e_{i_1} \otimes \cdots \otimes e_{i_\rho}$.

In this formulation, $\Gamma = (g, n, \beta, \rho, \mu)$ with $\mu = (\mu_1, \ldots, \mu_\rho) \in \mathbb{N}^\rho$ a partition of the intersection number $(\beta.E) = |\mu| := \sum_{i=1}^\rho \mu_i$. For $A \in H^*(Y)^{\otimes n}$ and $\varepsilon \in H^*(E)^{\otimes\rho}$, the relative invariant of stable maps with topological type Γ (i.e. with contact order μ_i in E at the i-th contact point) is

$$\langle A \mid \varepsilon, \mu\rangle_\Gamma^{(Y,E)} := \int_{[\overline{M}_\Gamma(Y,E)]^{virt}} e_Y^* A \cup e_E^*\varepsilon$$

where $e_Y : \overline{M}_\Gamma(Y, E) \to Y^n$, $e_E : \overline{M}_\Gamma(Y, E) \to E^\rho$ are evaluation maps on marked points and contact points respectively.

If $\Gamma = \coprod_\pi \Gamma^\pi$, the relative invariants (with disconnected domain curves)

$$\langle A \mid \varepsilon, \mu\rangle_\Gamma^{\bullet(Y,E)} := \prod\nolimits_\pi \langle A \mid \varepsilon, \mu\rangle_{\Gamma^\pi}^{(Y,E)}$$

are defined to be the product of the connected components.

An admissible triples $\eta = (\Gamma_1, \Gamma_2, I_\rho)$ consists of (possibly disconnected) topological types

$$\Gamma_i = \coprod\nolimits_{\pi=1}^{|\Gamma_i|} \Gamma_i^\pi$$

with the same contact order partition μ under the identification I_ρ of contact points. The gluing $\Gamma_1 +_{I_\rho} \Gamma_2$ has type (g, n, β) and is connected. We denote by Ω the equivalence class of all admissible triples, also by Ω_β and Ω_μ the subset with fixed degree β and fixed contact order μ respectively.

4.3. Reduction to relative local models

First notice that $A_1(\tilde{E}) = \iota_{2*}A_1(E)$ since both are projective bundles over Z. We then have

$$\phi^*\beta = \beta_1 + \beta_2$$

by regarding β_2 as a class in $E \subset Y$.

For the n-point function $\langle\alpha\rangle^X = \sum_{\beta\in NE(X)}\langle\alpha\rangle^X_\beta\, q^\beta$ we have

$$\begin{aligned}\langle\alpha\rangle^X &= \sum_{\beta\in NE(X)}\sum_{\eta\in\Omega_\beta}\sum_I C_\eta\langle\alpha_1 \mid e_I,\mu\rangle^{\bullet(Y_1,E)}_{\Gamma_1}\langle\alpha_2 \mid e^I,\mu\rangle^{\bullet(Y_2,E)}_{\Gamma_2}\, q^{\phi^*\beta}\\ &= \sum_\mu\sum_I\sum_{\eta\in\Omega_\mu} C_\eta\left(\langle\alpha_1 \mid e_I,\mu\rangle^{\bullet(Y_1,E)}_{\Gamma_1}\, q^{\beta_1}\right)\left(\langle\alpha_2 \mid e^I,\mu\rangle^{\bullet(Y_2,E)}_{\Gamma_2}\, q^{\beta_2}\right).\end{aligned}$$

To simplify the generating series, we consider also absolute invariants $\langle\alpha\rangle^{\bullet X}$ with possibly disconnected domain curves as before. Then by comparing the order of automorphisms,

$$\langle\alpha\rangle^{\bullet X} = \sum_\mu m(\mu)\sum_I\langle\alpha_1 \mid e_I,\mu\rangle^{\bullet(Y_1,E)}\langle\alpha_2 \mid e^I,\mu\rangle^{\bullet(Y_2,E)}$$

where the generating series with possibly disconnected domain curves are

$$\langle A \mid \varepsilon,\mu\rangle^{\bullet(\tilde{E},E)} := \sum_{\Gamma;\,\mu_\Gamma=\mu}\frac{1}{|\mathrm{Aut}\,\Gamma|}\langle A \mid \varepsilon,\mu\rangle^{\bullet(\tilde{E},E)}_\Gamma\, q^{\beta^\Gamma}.$$

To compare $\mathcal{F}\langle\alpha\rangle^{\bullet X}$ and $\langle\mathcal{F}\alpha\rangle^{\bullet X'}$, by the cohomology reduction we may assume that $\alpha_1 = \alpha_1'$ and $\alpha_2' = \mathcal{F}\alpha_2$. By comparing with the similar expression for $\langle\mathcal{F}\alpha\rangle^{\bullet X'}$, the relative terms for (Y,E) are identical. It remains to compare

$$\langle\alpha_2 \mid e^I,\mu\rangle^{\bullet(\tilde{E},E)} \quad\text{and}\quad \langle\mathcal{F}\alpha_2 \mid e^I,\mu\rangle^{\bullet(\tilde{E}',E)}.$$

We further split the sum into connected invariants.

$$\langle A \mid \varepsilon,\mu\rangle^{\bullet(\tilde{E},E)} = \sum_{P\in P(\mu)}\prod_{\pi\in P}\sum_{\Gamma^\pi}\frac{1}{|\mathrm{Aut}\,\mu^\pi|}\langle A^\pi \mid \varepsilon^\pi,\mu^\pi\rangle^{(\tilde{E},E)}_{\Gamma^\pi}\, q^{\beta^{\Gamma^\pi}}$$

where Γ^π is a connected part with contact order μ^π induced from μ and $P(\mu)$ is the set of all partitions $P : \mu = \sum_{\pi\in P}\mu^\pi$.

Notice that only β^{Γ^π} can vary in the sum over Γ^π and we may denote the generating series of connected relative invariants as sum over $\beta_2 \in NE(\tilde{E})$. This reduces the problem to $\langle A^\pi \mid \varepsilon^\pi,\mu^\pi\rangle$. We summarize the result as follows.

Proposition 14 *To prove $\mathcal{F}\langle\alpha\rangle^X \cong \langle\mathcal{F}\alpha\rangle^{X'}$, it is enough to show that*

$$\mathcal{F}\langle A \mid \varepsilon,\mu\rangle \cong \langle\mathcal{F}A \mid \varepsilon,\mu\rangle.$$

4.4. Relative invariants to absolute invariants

Inspired by a method of Maulik and Pandharipande [17], we further reduce the relative local cases to the absolute local cases with at most descendent insertions along E.

Proposition 15 *For simple ordinary flops $\tilde{E} \dashrightarrow \tilde{E}'$, to prove*

$$\mathcal{F}\langle A \mid \varepsilon, \mu\rangle \cong \langle \mathcal{F}A \mid \varepsilon, \mu\rangle$$

for any A and (ε, μ), it is enough to show that

$$\mathcal{F}\langle A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho\rangle \cong \langle \mathcal{F}A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho\rangle$$

for any possible insertions $A \in H^*(\tilde{E})^{\oplus n}$, $k_j \in \mathbb{N} \cup \{0\}$ and $\varepsilon_j \in H^*(E)$.

We apply the deformation to the normal cone for $Z \hookrightarrow \tilde{E}$ to get $W \to \mathbb{A}^1$. Then $W_0 = Y_1 \cup Y_2$ with $Y_1 \cong \mathbb{P}_E(\mathcal{O}_E(-1,-1) \oplus \mathcal{O})$ a $\mathbb{P}^1$ bundle and $Y_2 \cong \tilde{E}$. Denote $E_0 = E = Y_1 \cap Y_2$ and $E_\infty \cong E$ the infinity divisor of Y_1.

Given a relative invariant $\langle \alpha_1, \ldots, \alpha_n \mid \varepsilon, \mu\rangle$ on $(\tilde{E}, E)$, the idea is to analyze the degeneration formula for $\langle \alpha_1, \ldots, \alpha_n, \tau_{\mu_1 - 1}\varepsilon_1, \ldots, \tau_{\mu_\rho - 1}\varepsilon_\rho\rangle^{\tilde{E}}$ and to use induction on the triple $(|\mu|, n, \rho)$ in the lexicographical order with ρ in the reverse order. Since $\rho \le |\mu|$, it is clear that there are only finitely many triples of lower order. The proposition holds for those cases by the induction hypothesis.

For $\beta = d_1\ell + d_2\gamma \in NE(\tilde{E})$, $(c_1(\tilde{E}).\beta) = d_2(c_1(\tilde{E}).\gamma)$, hence by the virtual dimension counting d_2 is uniquely determined for a given generating series with fixed cohomology insertions.

Note that $NE(Y_1) = \mathbb{Z}_+\delta + \mathbb{Z}_+\gamma + \mathbb{Z}_+\bar{\gamma}$ with γ, δ the two line classes in E and $\bar{\gamma}$ the fiber class of Y_1 and $NE(Y_2) = \mathbb{Z}_+\ell \oplus \mathbb{Z}_+\gamma$. A curve class $\beta = d_1\ell + d_2\gamma \in NE(\tilde{E})$ is split into $\beta_1 = a\delta + b\gamma + c\bar{\gamma} \in NE(Y_1)$ and $\beta_2 = d\ell + e\gamma \in NE(Y_2)$ which satisfy

$$a, b, c, d, e \ge 0, \quad a + d = d_1, \quad c = d_2$$

and the total contact order condition

$$e = (\beta_2.E)_{Y_2} = (\beta_1.E)_{Y_1} = -a - b + c.$$

In particular, $e \le d_2$ with $e = d_2$ if and only if that $a = b = 0$. In this case $\beta_1 = d_2\bar{\gamma}$ and the invariants on (Y_1, E) are fiber class integrals.

Since $\varepsilon_i|_Z = 0$, one may choose the cohomology representative $\varepsilon_i(0) = (\iota_{1*}\varepsilon_i, 0)$. For a general cohomology insertion $\alpha \in H^*(\tilde{E})$, the representative can be chosen to be $\alpha(0) = (a, \alpha)$ for some a.

As before the relative invariants on (Y_1, E) can be regarded as constants under $\mathcal{F}$. Let $(\varepsilon_1, \ldots, \varepsilon_\rho) = e_I = (e_{i_1}, \ldots, e_{i_\rho})$. Then

$$\langle \alpha_1, \ldots, \alpha_n, \tau_{\mu_1 - 1} e_{i_1}, \ldots, \tau_{\mu_\rho - 1} e_{i_\rho} \rangle^{\bullet \tilde{E}} = \sum_{\mu'} m(\mu') \times$$
$$\sum_{I'} \langle \tau_{\mu_1 - 1} e_{i_1}, \ldots, \tau_{\mu_\rho - 1} e_{i_\rho} \mid e^{I'}, \mu' \rangle^{\bullet (Y_1, E)} \langle \alpha_1, \ldots, \alpha_n \mid e_{I'}, \mu' \rangle^{(\tilde{E}, E)} + R,$$

where R denotes the remaining terms which either have total contact order smaller than d_2 or have number of insertions fewer than n on the $(\tilde{E}, E)$ side or the invariants on $(\tilde{E}, E)$ are disconnected ones.

The crucial point is that we can show that the highest order term in the sum consists of the single term

$$C(\mu) \langle \alpha_1, \ldots, \alpha_n \mid e_I, \mu \rangle^{(\tilde{E}, E)}$$

where $C(\mu) \neq 0$. Then the induction hypothesis for R together with the absolute cases with at most descendent insertions along E give us the desired relative case.

4.5. Examples

I would like to illustrate the degeneration formula by treating a slightly more general case which includes *simple* (r, r') *flips*. Consider $(\psi, \bar{\psi}) : (X, Z) \to (\bar{X}, S)$ a log-extremal contraction as before. ψ is an ordinary (r, r') flipping contraction if

(i) $Z = \mathbb{P}_S(F)$ for some rank $r+1$ vector bundle F over S,
(ii) $N_{Z/X}|_{Z_s} \cong \mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus (r'+1)}$ for each $\bar{\psi}$-fiber Z_s, $s \in S$.

The construction of the (r, r') flip $f : X \dashrightarrow X'$ is the same as the $\mathbb{P}^r$ flop case.

We will consider simple (r, r') flips with K_X nef. If $\beta = d\ell$ then the invariant depends only on Z, $\alpha|_Z$ and $N_{Z/X}$. In particular $\langle \alpha \rangle^X_{g,n,d\ell} = \langle p^*(\alpha|_Z) \rangle^{\tilde{E}}_{g,n,d\ell}$. For other curve classes, we will see that the invariants *degenerate cleanly*. For odd dimensional classes they must contribute on Y_1 since $H^*(Y_2)$ has only algebraic classes, for the purpose of comparing GW invariants we thus consider only $\alpha_i \in H^{2l_i}(X)$. Because of the divisor axiom, we require also that $l_i \geq 2$ for all i.

The following proposition works for more general setup:

Proposition 16 *Let $\phi : Y \to X$ be the blow-up of X along a smooth center Z of dimension r and codimension $r'+1$ with K_X nef and $r \leq r'+1$. Then*

(1) $C_\eta \neq 0$ only if $g_1 = 0$, $v_1 = \mu = \rho \neq 0$ and $\mu_1 \equiv 1$, $v_2 = 1$.

(2) If $r \le 2$ then

$$\langle\alpha\rangle^X_{g,n,\beta} = \langle\tilde\alpha\rangle^Y_{g,n,\phi^*\beta} + \sum_{\eta,\,\rho_\eta\ne 0} C_\eta\,\langle -\rangle^{(Y,E)}_{\Gamma_1}\cdot\langle p^*(\alpha|_Z)\rangle^{(\tilde E,E)}_{\Gamma_2}.$$

The sum over η is trivial for $r = 1$. For $r = 2$ (so $l_i = 2$ for all i), the relative invariants are zero dimensional on (Y, E) and top dimensional on $(\tilde E, E)$.

(3) Let $f : X \dashrightarrow X'$ be a simple $(2, r')$ flip with K_X nef (so $r' \ge 2$). Let $\beta \notin \mathbb{Z}\ell$. If $\beta_2 = \lambda\ell + \mu\gamma \in NE(\tilde E) = \mathbb{R}_+\ell \oplus \mathbb{R}_+\gamma$ with γ the fiber line class of $\tilde E \to Z$, then the only possible values for (g, n, λ, r') with non-trivial sum are

$(0, 0, 2, 4)$, $(0, 0, 3, 3)$, $(0, 1, 2, 3)$, $(0, 2, 1, r')$, $(0, 2, \lambda, 2)$ *and* $(1, 0, \lambda, 2)$.

Proof. For $\eta = (\Gamma_1, \Gamma_2, I)$ associated to the topological type (g, n, β), let d, d_{Γ_1} and d_{Γ_2} be the virtual dimension (without marked points) of stable morphisms into X and relative stable morphisms into (Y_1, E), (Y_2, E) of corresponding admissible graph respectively. We have $l_1 + \cdots + l_n = d + n$. Moreover, since $\dim E = r + r'$, the gluing structure of virtual moduli cycles implies that $d = d_{\Gamma_1} + d_{\Gamma_2} - (r + r')\rho$.

We assume that the summand given by η is not zero. Since $\beta \ne d\ell$ and $A_1(Y_2)$ is spanned by ℓ and a fiber line γ, we see that $\beta_1 \ne 0$ and $\Gamma_1 \ne \emptyset$. If $\rho = 0$ then $\Gamma_2 = \emptyset$ by the connectedness of graph, and this gives the first term of the formula. So we assume that $\rho \ne 0$. By reordering, we may assume that in the degeneration expression α_i appears in the Y_1 part for $1 \le i \le m$ and α_i appears in the Y_2 part for $m + 1 \le i \le n$. By transversality, the corresponding relative invariant is non-trivial only if $2 \le l_i \le r$ for $m + 1 \le i \le n$. If $r = 1$ this simply means that all α_i's appear in Y_1.

For each $v \in V_{\Gamma_1}$, the virtual dimension $d_{\Gamma_1}(v)$ is given by

$$c_1(Y).\beta_1(v) + (r + r' + 1)(1 - g_1(v)) + (3g_1(v) - 3) + \sum_{w\in R_{\Gamma_1},\, w\mapsto v} (1 - \mu_1(w)).$$

Denote by ρ_v, μ_v and n_v the number of roots, the total contact order and the number of marked points along v respectively.

¿From $c_1(Y) = -K_Y = -\phi^*K_X - r'E$ and $(E.\beta_1(v))_{Y_1} = \mu_v$, we get

$$\begin{aligned} d_{\Gamma_1}(v) &= -(K_Y.\beta_1(v)) + (1 - g_1(v))(r + r' - 2) + \rho_v - \mu_v \\ &= -(K_X.\phi_*\beta_1(v)) - (r' + 1)\mu_v + (r + r' - 2) + \rho_v - g_1(v)(r + r' - 2). \end{aligned}$$

The relative invariant on Y_1 along v is non-trivial only if

$$2n_v \le \sum_{w\in L_{\Gamma_1},\, w\mapsto v} l_{I_1(w)} \le d_{\Gamma_1}(v) + n_v.$$

In particular, we must have $d_{\Gamma_1}(v) \geq n_v \geq 0$.

We claim that $\rho_v = \mu_v = 1$ for all v. Indeed if $\mu_v \geq 2$ then the nefness of K_X implies $d_{\Gamma_1}(v) \leq -(\mu_v - \rho_v) - (\mu_v - 2)r' + (r - r' - 2) \leq -1$ which leads to a contradiction. Thus $\mu_v = 1$, which implies that $\rho_v = 1$ and

$$d_{\Gamma_1}(v) \leq -(K_X.\phi_*\beta_1(v)) + (r-2) - (r + r' - 2)g_1(v).$$

Since K_X is nef, for any r it is clear that $g_1(v) \geq 1$ leads to $d_{\Gamma_1}(v) < 0$, hence $g_1(v) = 0$. By the connectedness of $\Gamma_1 + \Gamma_2$, we must have $v_2 = 1$, $g_2 = g$ and the contact order for each root is one. This proves (1).

For $r = 1$, the above formula leads to $d_{\Gamma_1} < 0$. This contradicts to $m \geq 0$, so the case $\rho \neq 0$ does not occur.

If $r = 2$ and $C_\eta \neq 0$ then $(K_X.\phi_*\beta_1(v)) = 0$ too. Moreover, $d_{\Gamma_1} = 0$, $m = 0$ and $l_i = 2$ for all $1 \leq i \leq n$. In this case $2n = d + n \Rightarrow d = n$. So

$$d_{\Gamma_2} = n + (r'+2)\rho \quad \text{and} \quad (d_{\Gamma_2} + n) - \sum\nolimits_{i=1}^n l_i = (r'+2)\rho = \dim E^\rho.$$

That is, the relative invariants are of the said dimensions. This proves (2).

For (3), it remains to classify the cases with nontrivial sum over η.

In the case of $(2, r')$ flips it gives $c_1(Y_2) = (r'+2)E - (r'-2)p^*h$. Since $(E.\beta_2)_{Y_2} = \mu$ and $(\tilde{h}.\beta_2)_{Y_2} = \lambda$ for $\beta_2 = \lambda\ell + \mu\gamma$, we get

$$\begin{aligned} d_{\Gamma_2} &= c_1(\tilde{E}).\beta_2 + (r'+3)(1-g) + 3g - 3 \\ &= (r'+2)\mu - (r'-2)\lambda + r'(1-g). \end{aligned}$$

This holds if and only if that $(d_{\Gamma_2} + n) - 2n = (r'+2)\mu$, that is $n + (r'-2)\lambda = r'(1-g)$. For any given (g, n, β), this equation for λ has at most one solution if $r' \neq 2$. If $r' > 2$ then since $\lambda > 0$ we must have $g = 0$. So $\lambda = (r'-n)/(r'-2)$ and then $n \leq 2$. If $r' = 2$ then λ is free.

¿From this, it is straightforward to write down all such (g, n, λ, r') as listed and the proof is completed. □

Remark 17 *Case (1) of the proposition is exactly Proposition 4.13 in [8] (where the proof was left to the readers). In fact it applies to situations other than simple flips. For example it applies to $(r-s, r')$ flips over an s dimensional base S. If $r = r'+1$, it applies to $\mathbb{P}^{r'}$ flops over an one dimensional base.*

5. GW invariants on local models

The basic strategy here is similar to the proof of Theorem 3.1. We start with one point invariants on toric varieties and then use induction together with reconstruction procedure to achieve our final result.

5.1. One-point functions on local models

Following Givental, we define a generating function of one point invariants with descendents

$$
\begin{aligned}
J_X := J_X(q, z^{-1}) &:= \sum_{\beta \in NE(X)} q^\beta J_X(\beta, z^{-1}) \in H^*(X)[\![z^{-1}]\!][\![q]\!] \\
&:= \sum_{\beta \in NE(X)} q^\beta e^X_{1*}\left(\frac{1}{z(z-\psi)} \cap [\overline{M}_{0,1}(X,\beta)]^{virt}\right).
\end{aligned}
$$

The toric data for the local model $X = \mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(-1)^{\oplus(r+1)} \oplus \mathcal{O})$ allows us to apply the known results of [5] [14] directly. For an effective curve class $\beta = d_1\ell + d_2\gamma$,

$$
J_X(\beta, z^{-1}) = P_\beta := \frac{\prod\limits_{m=-\infty}^{0} (\xi - h + mz)^{r+1}}{\prod\limits_{m=1}^{d_1} (h+mz)^{r+1} \prod\limits_{m=-\infty}^{d_2-d_1} (\xi - h + mz)^{r+1} \prod\limits_{m=1}^{d_2} (\xi + mz)}
$$

without change of variables ("mirror transformation") due to the uniqueness theorem and the fact that $P_\beta = O(1/z^2)$ in $1/z$ power series expansion.

The cohomology (Chow ring) is given by

$$
H^*(X) = A^*(X) = \mathbb{Z}[h, \xi]/(h^{r+1}, (\xi - h)^{r+1}\xi)
$$

where h is the pull-back of the hyperplane class in Z and ξ is the infinity divisor of X. We have that $\mathcal{F}\ell = -\ell'$, $\mathcal{F}\gamma = \ell' + \gamma'$, $\mathcal{F}\xi = \xi'$ and $\mathcal{F}h = \xi' - h'$.

The key observation on P_β is that if $d_2 - d_1 < 0$ then the middle factor in the denominator of P_β goes to the numerator instead which has a factor $(\xi - h)^{r+1}$. Thus it vanishes after multiplication by ξ.

For $d_2 \geq d_1$, since $J_X = \sum_{\beta \in NE(X)} q^\beta P_\beta$, we can get by direct computation

$$
\mathcal{F}(J_X \xi.\alpha) = J_{X'}\xi'.\mathcal{F}\alpha
$$

and hence the important functional equation

$$
\mathcal{F}\left\langle \tau_k \xi.\alpha \right\rangle^X = \left\langle \tau_k \xi'.\mathcal{F}\alpha \right\rangle^{X'}.
$$

This together with the functional equation for extremal rays form the two generators of the general functional equations discussed in next section.

Note that the virtual dimension of an n-point invariants in degree $\beta = d_1\ell + d_2\gamma$ is given by $D_{n,\beta} = (r+2)d_2 + 2r + n - 2$, so for a fixed set of cohomology insertions there could be at most one d_2 supporting non-trivial invariants and for the corresponding n-point function the summation over d_2 is unnecessary. Here we find that $\langle \tau_k \xi \alpha \rangle$ is a finite sum and $\mathcal{F} \langle \tau_k \xi . \alpha \rangle = \langle \tau_k \xi' . \mathcal{F} \alpha \rangle$ holds without the need of analytic continuation.

5.2. The functional equations in general

The compatibility of functional equations under the reconstruction procedure is proved with help from operators δ_H's which generalize $q^\ell d/dq^\ell$, the one used in the proof for invariance of big quantum product restricted to exceptional curve classes.

For a power series $f = \sum_\beta a_\beta \, q^\beta$ and a divisor H, we define the operator

$$\delta_H f := \sum_\beta (H.\beta) a_\beta \, q^\beta = \Big((H.\ell) q^\ell \frac{\partial}{\partial q^\ell} + (H.\gamma) q^\gamma \frac{\partial}{\partial q^\gamma} \Big) f.$$

We can formalizes the argument in the proof of §3.4 as follows.

The differential operator δ_H is $\mathcal{F}$ equivariant. That is,

$$\mathcal{F} \circ \delta_H = \delta_{\mathcal{F} H} \circ \mathcal{F}.$$

In particular, if $\mathcal{F} \langle \alpha \rangle \cong \langle \mathcal{F} \alpha \rangle$ then $\mathcal{F} \delta_H \langle \alpha \rangle \cong \delta_{\mathcal{F} H} \langle \mathcal{F} \alpha \rangle$ too.

Write $\beta = d_1 \ell + d_2 \gamma$. If $d_2 = 0$, the whole setting on Gromov-Witten invariants goes back to quantum corrections attached to the extremal ray $\mathbb{Z}\ell$. For general d_2, the following theorem is our final result.

Theorem 18 *Let $\langle \alpha \rangle = \langle \alpha_1, \ldots, \alpha_n \rangle$ with $\alpha_i \in H^*(X) \cup \tau_\bullet H^*(E)$. If $d_2 \neq 0$ then*

$$\mathcal{F} \langle \alpha \rangle \cong \langle \mathcal{F} \alpha \rangle.$$

One basic fact worth mentioning once more is that by the virtual dimension count, each set of insertions can support at most one d_2. Let $d_2 \geq 1$ and $n \geq 2$. We may and will make one more assumption that ξ appears in some α_i. If not, then there will be no descendent insertions and by the divisor axiom, we may write

$$\langle \alpha_1, \ldots, \alpha_n \rangle = \langle \alpha_1, \ldots, \alpha_n, \xi \rangle / d_2.$$

Also, by reordering we may assume that $\alpha_n = \tau_s \xi a$, $s \geq 0$. Write $\alpha_1 = \tau_k h^l \xi^j$. The induction procedure is to move divisors in α_1 into α_n in the order of ψ, h and ξ. That is we use induction on the following five numbers in the alphabetical order:

$$(d_2, n, k, l, j).$$

Step1. For ψ we use equation $\psi_1 = -\psi_n + [D_{1|n}]^{virt}$. If $k \geq 1$ then $j \neq 0$ and we get

$$\begin{aligned}\langle \tau_k h^l \xi^j, \ldots, \tau_s \xi a\rangle &= -\langle \tau_{k-1} h^l \xi^j, \ldots, \tau_{s+1}\xi a\rangle \\ &\quad + \sum_i \langle \tau_{k-1} h^l \xi^j, \ldots, T_i\rangle\langle T^i, \ldots, \tau_s \xi a\rangle.\end{aligned}$$

For each i, if one of d_2^L and d_2^R is zero then since both terms contain ξ classes the splitting term must vanish. So we may assume that $d_2^L < d_2$ and $d_2^R < d_2$ and these terms are done by the induction hypothesis. By performing this procedure to $\alpha_1, \ldots, \alpha_{n-1}$ we may assume that the only descendent insertion is α_n.
Step 2. For h, if $l \geq 2$ or $l = 1$ but $j \neq 0$ we use the divisor relation to get

$$\begin{aligned}\langle h^l \xi^j, \ldots, \tau_s \xi a\rangle &= \langle h^{l-1}\xi^j, \ldots, \tau_s \xi a h\rangle + \delta_h \langle h^{l-1}\xi^j, \ldots, \tau_{s+1}\xi a\rangle \\ &\quad - \sum_i \delta_h \langle h^{l-1}\xi^j, \ldots, T_i\rangle\langle T^i, \ldots, \tau_s \xi a\rangle.\end{aligned}$$

The only case for the splitting term to have one factor to have the same d_2 and n is of the form

$$\delta_h \langle h^{l-1}\xi^j, T_i\rangle\langle T^i, \alpha_2, \ldots, \alpha_{n-1}, \tau_s \xi a\rangle,$$

where the two-point invariant has $d_2^L = 0$. But then $l - 1 < r$ forces it to vanish.

By induction we are left with the case $\alpha_1 = h$. The divisor axiom implies that

$$\langle h, \ldots, \tau_s \xi a\rangle = \delta_h \langle \ldots, \tau_s \xi a\rangle + \langle \ldots, \tau_{s-1}\xi a h\rangle.$$

Since both terms have one less marked points, they are done by induction.
Step 3. For ξ, the argument is entirely similar. For $j \geq 2$, the divisor relation says that

$$\begin{aligned}\langle \xi^j, \ldots, \tau_s \xi a\rangle &= \langle \xi^{j-1}, \ldots, \tau_s \xi^2 a\rangle + \delta_\xi \langle \xi^{j-1}, \ldots, \tau_{s+1}\xi a\rangle \\ &\quad - \sum_i \delta_\xi \langle \xi^{j-1}, \ldots, T_i\rangle\langle T^i, \ldots, \tau_s \xi a\rangle.\end{aligned}$$

We then have $d_2^L < d_2$ and $d_2^R < d_2$ as before. If $j = 1$ we get

$$\langle \xi, \ldots, \tau_s \xi a\rangle = \delta_\xi \langle \ldots, \tau_s \xi a\rangle + \langle \ldots, \tau_{s-1}\xi^2 a\rangle$$

and both terms have fewer marked points. The result follows.

Practically the above inductive procedure leads to explicit determination of GW invariants, though the computations are somewhat tedious. In the end, I would like to give an example which consists of all precise computations and guides readers through the difficulties.

5.3. A typical example

Descendent invariants for simple $\mathbb{P}^2$ flop with $d_2 = 1$ and $n = 3$. The virtual dimension is $D_{3,\beta} = (2+2) \times 1 + 2 \times 2 + 3 - 2 = 9$.

1. To determine $\langle h^2, h^2, \tau_4 \xi \rangle$:

Recall the divisor relation

$$e_i^* L = e_j^* L + \sum_{\beta_1+\beta_2=\beta} ((\beta_2.L)[D_{ik,\beta_1|j,\beta_2}]^{virt} - (\beta_1.L)[D_{i,\beta_1|jk,\beta_2}]^{virt}).$$

For simplicity, we write $\langle h^2, h^2, \tau_4\xi\rangle_{d_1,d_2}$ instead of $\langle h^2, h^2, \tau_4\xi\rangle_{d_1\ell+d_2\gamma}$. Then

$$\begin{aligned}\langle h^2, h^2, \tau_4\xi\rangle_{d_1,1} = \sum_d & (d_1 - d)\langle h, \tau_4\xi, T_i\rangle_{d,1}\langle T^i, h^2\rangle_{d_1-d,0} \\ & - d\langle h, T_i\rangle_{d,0}\langle T^i, h^2, \tau_4\xi\rangle_{d_1-d,1}.\end{aligned}$$

Note that for the first term any addition to the power of $e_2^* h^2$ leads to zero and for $d_2 = 0$, since $\xi|_Z = 0$ we get trivial invariant if one of the insertions involves ξ. Now we fix a cohomology basis $\{T_i\}$ and its dual basis $\{T^i\}$ as follows.

$$\begin{aligned}\{(T_i, T^i)\} = \{&(X, h^2\xi^3), (\xi, h^2\xi^2), (\xi^2, h^2\xi), (\xi^3, h^2), (h, h\xi^3 - 3h^2\xi^2), \\ &(h\xi, h\xi^2 - 3h^2\xi), (h\xi^3, h), (h^2, \xi^3 + 3h^2\xi - 3h\xi^2), \\ &(h^2\xi, \xi^2 + 3h^2 - 3h\xi), (h^2\xi^2, \xi - 3h), (h^2\xi^3, X)\}\end{aligned}$$

Here $h^3 = 0$ and $(\xi - h)^3\xi = 0$.

If $d_1 - d = 0$, then the only possible non-trivial term is

$$\langle h, h\xi^3\rangle_{0,0}\langle h, h^2, \tau_4\xi\rangle_{0,1}.$$

However in this case, the coefficient $d = 0$.

For $d_1 - d > 0$, the only non-trivial contribution for $\langle T^i, h^2\rangle_{d_1-d,0}$ is

$$\langle h^2, h^2\rangle_{d_1-d,0} = (-1)^{(d_1-d-1)(2+1)}\frac{1}{d_1-d}$$

and the corresponding term for $\langle h, \tau_4\xi, T_i\rangle_{d,1}$ is

$$\begin{aligned}\langle h, \tau_4\xi, \xi^3 + 3h^2\xi - 3h\xi^2\rangle_{d,1} = {} & d\langle \tau_4\xi, \xi^3 + 3h^2\xi - 3h\xi^2\rangle_{d,1} \\ & + \langle \tau_3\xi h, \xi^3 + 3h^2\xi - 3h\xi^2\rangle_{d,1}\end{aligned}$$

by divisor axiom. Each term of the right side is determined as follows.

$$\langle \xi^3, \tau_4\xi\rangle_{d,1} = \langle \xi^2, \tau_4\xi^2\rangle_{d,1} + \langle \xi^2, \tau_5\xi\rangle_{d,1} - \sum \langle \xi^2, T_i\rangle_{d,0}\langle T^i, \tau_4\xi\rangle_{0,1}$$

and $\langle \xi^2, T_i\rangle_{d,0} = 0$, so $\langle \xi^3, \tau_4\xi\rangle_{d,1} = \langle \xi^2, \tau_4\xi^2\rangle_{d,1} + \langle \xi^2, \tau_5\xi\rangle_{d,1}$. Similarly

$$\begin{aligned}\langle \xi^3, \tau_4\xi\rangle_{d,1} &= \langle \xi, \tau_4\xi^3\rangle_{d,1} + \langle \xi, \tau_5\xi^2\rangle_{d,1} + \langle \xi, \tau_5\xi^2\rangle_{d,1} + \langle \xi, \tau_6\xi\rangle_{d,1} \\ &= \langle \tau_4\xi^3\rangle_{d,1} + \langle \tau_3\xi^4\rangle_{d,1} + 2\langle \tau_5\xi^2\rangle_{d,1} + 2\langle \tau_4\xi^3\rangle_{d,1} \\ &\quad + \langle \tau_6\xi\rangle_{d,1} + \langle \tau_5\xi^2\rangle_{d,1}\end{aligned}$$

In §5.1, we have introduced a recipe to compute one-point descendent invariants. On one hand, (only non-trivial for $d_2 \geq d_1$)

$$J_X(\beta, z^{-1})\,\xi.\alpha = \frac{\xi\alpha}{\prod_{m=1}^{d_1}(h+mz)^{r+1}\prod_{m=1}^{d_2-d_1}(\xi-h+mz)^{r+1}\prod_{m=1}^{d_2}(\xi+mz)}.$$

In this case,

$$J_X(d\ell+\gamma, z^{-1})\,\xi.\alpha = \frac{\xi\alpha}{(\xi+z)\prod_{m=1}^{d}(h+mz)^{r+1}\prod_{m=1}^{1-d}(\xi-h+mz)^{r+1}}.$$

On the other hand, by the definition of $J_X(d\ell+\gamma, z^{-1})$,

$$J_X(d\ell+\gamma, z^{-1})\xi.\alpha = \sum_{k\geq 0}\frac{\langle \tau_k\xi\alpha\rangle_{d,1}}{z^{k+2}}.$$

For $d = 0$, the expansion of $J_X(d\ell+\gamma, z^{-1})$ is

$$\begin{aligned}&\frac{1}{z^4} + \frac{-4\xi+3h}{z^5} + \frac{10\xi^2-15\xi h+6h^2}{z^6} + \frac{-20\xi^3+45\xi^2 h-36\xi h^2+10h^3}{z^7} \\ &\quad + \frac{35\xi^4-105\xi^3 h+126\xi^2h^2-70\xi h^3+15h^4}{z^8} + O(1/z^9).\end{aligned}$$

By comparing the coefficients of $\dfrac{1}{z^5}$, we obtain

$$\begin{aligned}\langle \tau_3\xi^4\rangle_{0,1} &= (-4\xi+3h)\xi^4 = (-4\xi+3h)(3\xi^3h-3\xi^2h^2) \\ &= -12(3\xi^3h-3\xi^2h^2)h + 12 + 9 = -15.\end{aligned}$$

Similarly we can determine all involved one point invariants and get

$$\begin{aligned}\langle \xi^3, \tau_4\xi\rangle_{0,1} &= \langle \tau_4\xi^3\rangle_{0,1} + \langle \tau_3\xi^4\rangle_{0,1} + 2\langle \tau_5\xi^2\rangle_{0,1} + 2\langle \tau_4\xi^3\rangle_{0,1} \\ &\quad + \langle \tau_6\xi\rangle_{0,1} + \langle \tau_5\xi^2\rangle_{0,1} \\ &= 21 + (-15) + 2\times(-21) + 2\times 21 + 21 - 21 = 6.\end{aligned}$$

For $d=1$, the expansion of $J_X(d\ell+\gamma, z^{-1})$ is

$$\frac{1}{z^4}+\frac{-3h-\xi}{z^5}+\frac{6h^2+(3h+\xi)\xi}{z^6}+\frac{-10h^3+(-6h^2-3\xi h-\xi^2)\xi}{z^7}$$
$$+\frac{15h^4+(10h^3+6\xi h^2+3\xi^2 h+\xi^3)\xi}{z^8}+O(1/z^9).$$

By comparing the coefficients of $\dfrac{1}{z^8}$, we obtain

$$\langle\tau_6\xi\rangle_{1,1}=(6\xi h^2+3\xi^2 h+\xi^3)\xi^2=6+9+6=21.$$

Similarly we can determine all involved one point invariants and get

$$\begin{aligned}\langle\xi^3,\tau_4\xi\rangle_{1,1}&=\langle\tau_4\xi^3\rangle_{1,1}+\langle\tau_3\xi^4\rangle_{1,1}+2\langle\tau_5\xi^2\rangle_{1,1}+2\langle\tau_4\xi^3\rangle_{1,1}\\&\quad+\langle\tau_6\xi\rangle_{1,1}+\langle\tau_5\xi^2\rangle_{1,1}\\&=21+(-15)+2\times(-21)+2\times 21+21-21=6.\end{aligned}$$

Repeating the same procedure, we get

$$\begin{aligned}&\langle h^2\xi,\tau_4\xi\rangle_{0,1}=6,\quad \langle h^2\xi,\tau_4\xi\rangle_{1,1}=0;\\&\langle h\xi^2,\tau_4\xi\rangle_{0,1}=6,\quad \langle h\xi^2,\tau_4\xi\rangle_{1,1}=0;\\&\langle\xi^3,\tau_3\xi h\rangle_{0,1}=6,\quad \langle\xi^3,\tau_3\xi h\rangle_{1,1}=-3;\\&\langle h^2\xi,\tau_3\xi h\rangle_{0,1}=0,\quad \langle h^2\xi,\tau_3\xi h\rangle_{1,1}=0;\\&\langle h\xi^2,\tau_3\xi h\rangle_{0,1}=3,\quad \langle h\xi^2,\tau_3\xi h\rangle_{1,1}=0.\end{aligned}$$

and thus

$$\begin{aligned}&\langle h,\tau_4\xi,\xi^3+3h^2\xi-3h\xi^2\rangle_{0,1}=6-9=-3,\\&\langle h,\tau_4\xi,\xi^3+3h^2\xi-3h\xi^2\rangle_{1,1}=6-3=3.\end{aligned}$$

Consequently,

$$\begin{aligned}\langle h^2,h^2,\tau_4\xi\rangle_{d_1,1}&=\sum_d(d_1-d)\langle h,\tau_4\xi,T_i\rangle_{d,1}\langle T^i,h^2\rangle_{d_1-d,0}\\&=d_1\times(-3)\times\frac{(-1)^{3(d_1-1)}}{d_1}+(d_1-1)\times 3\times\frac{(-1)^{3(d_1-1-1)}}{d_1-1}\\&=(-1)^{d_1}\times 6\qquad(\text{for } d_1>1)\end{aligned}$$

and equals -3 for $d_1=1$. Hence let $q_1=q^\ell$ and $q_2=q^\gamma$. We have

$$\begin{aligned}\langle h^2,h^2,\tau_4\xi\rangle&=-3q_1q_2+\sum_{d_1>1}(-1)^{d_1}\times 6q_1^{d_1}q_2\\&=3q_1q_2-6\frac{q_1q_2}{1+q_1}.\end{aligned}$$

2. Applying the same recipe as above, we can obtain

$$\langle \xi^2, \xi^2, \tau_4\xi \rangle = 9q_2 + 9q_1q_2,$$

$$\langle h\xi, h\xi, \tau_4\xi \rangle = \langle h^2, \xi^2, \tau_4\xi \rangle = 3q_2,$$

$$\langle h\xi, h^2, \tau_4\xi \rangle = 0, \quad \langle h\xi, \xi^2, \tau_4\xi \rangle = 6q_2 + 3q_1q_2.$$

3. To show $\mathcal{F}\langle h^2, h^2, \tau_4\xi \rangle \cong \langle \mathcal{F}h^2, \mathcal{F}h^2, \mathcal{F}\tau_4\xi \rangle$.
On one hand,

$$\mathcal{F}\langle h^2, h^2, \tau_4\xi \rangle = 3q_1'^{-1}(q_1'q_2') - 6\frac{q_1'^{-1}(q_1'q_2')}{1+q_1'^{-1}} = 3q_2' - \frac{6q_1'q_2'}{1+q_1'}.$$

Here we use the correspondence $\mathcal{F}\ell = -\ell'$ and $\mathcal{F}\gamma = \ell' + \gamma'$.
On the other hand,

$$\begin{aligned}
\langle \mathcal{F}h^2, \mathcal{F}h^2, \mathcal{F}\tau_4\xi \rangle &= \langle (\xi' - h')^2, (\xi' - h')^2, \tau_4\xi' \rangle \\
&= \langle \xi'^2, \xi'^2, \tau_4\xi' \rangle + 4\langle \xi'h', \xi'h', \tau_4\xi' \rangle + \langle h'^2, h'^2, \tau_4\xi' \rangle \\
&\quad - 4\langle \xi'h', h'^2, \tau_4\xi' \rangle - 4\langle \xi'h', \xi'^2, \tau_4\xi' \rangle + 2\langle \xi'^2, h'^2, \tau_4\xi' \rangle \\
&= \big(9 + 9q_1' + 12 + 3q_1' - 6\frac{q_1'}{1+q_1'} - 24 - 12q_1' + 6\big)q_2' \\
&= \Big(3 - 6\frac{q_1'}{1+q_1'}\Big)q_2' = 3q_2' - \frac{6q_1'q_2'}{1+q_1'}.
\end{aligned}$$

Here we use the correspondence $\mathcal{F}\xi = \xi'$ and $\mathcal{F}h = \xi' - h'$. Similarly, we have the simpler verifications on the other five cases.

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