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Flops, motives and invariance of quantum rings

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Abstract

For ordinary flops, the correspondence defined by the graph closure is shown to give equivalence of Chow motives and to preserve the Poincaré pairing. In the case of simple ordinary flops, this correspondence preserves the big quantum cohomology ring after an analytic continuation over the extended Kähler moduli space.

For Mukai flops, it is shown that the birational map for the local models is deformation equivalent to isomorphisms. This implies that the birational map induces isomorphisms on the full quantum rings and all the quantum corrections attached to the extremal ray vanish.

0. Introduction

0.1. Statement of main results

Let X be a smooth complex projective manifold and $\psi : X \to \bar{X}$ a flopping contraction in the sense of minimal model theory, with $\bar{\psi} : Z \to S$ the restriction map on the exceptional loci. Assume that

(i) $\bar{\psi}$ equips Z with a $\mathbb{P}^r$-bundle structure $\bar{\psi} : Z = \mathbb{P}_S(F) \to S$ for some rank $r+1$ vector bundle F over a smooth base S,

(ii) $N_{Z/X}|_{Z_s} \cong \mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)}$ for each $\bar{\psi}$-fiber Z_s, $s \in S$.

It is not hard to see that the corresponding *ordinary* $\mathbb{P}^r$ *flop* $f : X \dashrightarrow X'$ exists. An ordinary flop is called *simple* if S is a point.

For a $\mathbb{P}^r$ flop $f : X \dashrightarrow X'$, the graph closure $[\bar{\Gamma}_f] \in A^*(X \times X')$ identifies the Chow motives $\hat{X}$ of X and $\hat{X}'$ of X'. Indeed, let $\mathcal{F} := [\bar{\Gamma}_f]$ then the transpose $\mathcal{F}^*$ is $[\bar{\Gamma}_{f^{-1}}]$. One has the following theorem.

Theorem 0.1. *For an ordinary* $\mathbb{P}^r$ *flop* $f : X \dashrightarrow X'$, *the graph closure* $\mathcal{F} := [\bar{\Gamma}_f]$ *induces* $\hat{X} \cong \hat{X}'$ *via* $\mathcal{F}^* \circ \mathcal{F} = \Delta_X$ *and* $\mathcal{F} \circ \mathcal{F}^* = \Delta_{X'}$. *In particular,* $\mathcal{F}$ *preserves the Poincaré pairing on cohomology groups.*

While the ring structure is in general not preserved under $\mathcal{F}$, the quantum cohomology ring is, when the analytic continuation on the Novikov variables is allowed.

THEOREM 0.2. *The big quantum cohomology ring is invariant under simple ordinary flops, after an analytic continuation over the extended Kähler moduli space.*

A contraction $(\psi, \bar{\psi}) : (X, Z) \to (\bar{X}, S)$ is of *Mukai type* if $Z = \mathbb{P}_S(F) \to S$ is a projective bundle under $\bar{\psi}$ and $N_{Z/X} = T^*_{Z/S}$. The corresponding algebraic flop $f : X \dashrightarrow X'$ exists and its local model can be realized as a *slice* of an ordinary flop. The following result is proved based upon our understanding of local geometry of Mukai flops.

THEOREM 0.3. *Let $f : X \dashrightarrow X'$ be a Mukai flop. Then X and X' are diffeomorphic, and have isomorphic Hodge structures and full Gromov–Witten theory. In fact, any local Mukai flop is a limit of isomorphisms and all quantum corrections attached to the extremal ray vanish.*

0.2. Motivations

This paper is the first of our study of the relationship between birational geometry and Gromov–Witten theory. Our motivations come from both fields.

K-equivalence in birational geometry

Two ($\mathbb{Q}$-Gorenstein) varieties X and X' are *K-equivalent* if there exist birational morphisms $\phi : Y \to X$ and $\phi' : Y \to X'$ with Y smooth such that

$$\phi^* K_X = \phi'^* K_{X'}.$$

K-equivalent smooth varieties have the same Betti numbers ([1] [25], see also [26] for a survey on recent development). However, the cohomology ring structures are in general different. Two natural questions arise here:

(1) Is there a *canonical* correspondence between the cohomology groups of K-equivalent smooth varieties?

(2) Is there a modified ring structure which is invariant under the K-equivalence relation?

The following conjecture was advanced by Y. Ruan [24] and the third author [26] in response to these questions.

CONJECTURE 0.4. *K-equivalent smooth varieties have canonically isomorphic quantum cohomology rings over the extended Kähler moduli spaces.*

The choice to start with ordinary flops is almost obvious. Ordinary flops are not only the first examples of K-equivalent maps, but also crucial to the general theory. In fact, one of the goals of this paper is to study some of their fundamental properties.

Functoriality in Gromov–Witten theory

In the Gromov–Witten theory, one is led to consider the problem of functoriality in quantum cohomology. Quantum cohomology is not functorial with respect to the usual operations: pull-backs, push-forwards, etc.. Y. Ruan [23] has proposed to study the *Quantum Naturality Problem*: finding the "morphisms" in the "category" of symplectic manifolds for which the quantum cohomology is "natural".

The main reason for lack of functoriality comes from the dimension count of the moduli of stable maps, where Gromov–Witten invariants are defined. (See §3.1 for the relevant definitions.) For example, given a birational morphism $f : Y \to X$, there is an induced morphism from moduli of maps to Y to moduli of maps to X. However, the (virtual) dimensions of the two moduli spaces are equal only if Y and X are K-equivalent. When the virtual dimensions of moduli spaces are different, the non-zero integral on moduli space of maps to X will be "pulled-back" to a zero integral on moduli space of maps to Y. Therefore, K-equivalence appears to be a necessary condition for this type of functoriality. Conjecture 0.4 suggests that the K-equivalence is also sufficient. We note here that there is of course no K-equivalent morphism between smooth varieties and a "flop-type" transformation is needed.

Theorem 0.2 can therefore be considered as establishing some functoriality of the genus zero Gromov–Witten theory in this direction. The higher genus case will be discussed in a separate paper.

Crepant resolution conjecture

Conjecture 0.4 can also be interpreted as a consistency check for the *Crepant Resolution Conjecture* [24] [3]. In general, there are more than one possible crepant resolution, but different crepant resolutions are K-equivalent. The consistency check naturally leads to a special version of Conjecture 0.4.

0.3. Contents of the paper

§1 studies the geometry of ordinary flops. The existence of ordinary flops is proved and explicit description of local models is given.

§2 is devoted to the correspondences and Chow motives of projective smooth varieties under an ordinary flop. The main result of this section is Theorem 0.1 alluded above. The ring structure is, however, not preserved. For a simple $\mathbb{P}^r$-flop, let h be the hyperplane class of $Z = \mathbb{P}^r$ and let $\alpha_i \in H^{2l_i}(X)$, with $l_i \le r$ and $l_1 + l_2 + l_3 = \dim X = 2r + 1$.

Proposition 0.5.

$$(\mathcal{F}\alpha_1.\mathcal{F}\alpha_2.\mathcal{F}\alpha_3) = (\alpha_1.\alpha_2.\alpha_3) + (-1)^r(\alpha_1.h^{r-l_1})(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3}).$$

For Calabi-Yau threefolds under a simple $\mathbb{P}^1$ flop, it is well known in the context of string theory (see e.g. [28]) that the defect of the classical product is exactly remedied by the quantum corrections attached to the extremal rays. This picture also emerged as part of Morrison's cone conjecture on birational Calabi-Yau threefolds [21] where Conjecture 0.4 for Calabi-Yau threefolds was proposed. For threefolds Conjecture 0.4 was proved by A. Li and Y. Ruan [15]. Their proof has three ingredients:

(1) A symplectic deformation and decomposition of K-equivalent maps into composite of ordinary $\mathbb{P}^1$ flops,

(2) the multiple cover formula for $\mathbb{P}^1 \cong C \subset X$ with $N_{C/X} \cong \mathcal{O}(-1)^{\oplus 2}$, and their main contribution:

(3) the theory of relative Gromov-Witten invariants and the degeneration formula.

In §3 a higher dimensional version of ingredient (2) is proved:

Theorem 0.6. *Let $Z = \mathbb{P}^r \subset X$ with $N_{Z/X} \cong \mathcal{O}(-1)^{r+1}$. Let ℓ be the line class in Z. Then for all $\alpha_i \in H^{2l_i}(X)$ with $1 \le l_i \le r$, $\sum_{i=1}^n l_i = 2r+1+(n-3)$ and $d \in \mathbb{N}$,*

$$\begin{aligned}\langle \alpha_1, \ldots, \alpha_n\rangle_{0,n,d} &\equiv \int_{[\overline{M}_{0,n}(X,d\ell)]^{virt}} e_1^*\alpha_1 \cdots e_n^*\alpha_n \\ &= (-1)^{(d-1)(r+1)} N_{l_1,\ldots,l_n} d^{n-3}(\alpha_1.h^{r-l_1})\cdots(\alpha_n.h^{r-l_n}).\end{aligned}$$

where $N_{l_1,\ldots,l_n}$ are recursively determined universal constants. $N_{l_1,\ldots,l_n}$ are independent of d and $N_{l_1,\ldots,l_n} = 1$ for $n = 2$ or 3. All other (primary) Gromov-Witten invariants with degree in $\mathbb{Z}\ell$ vanish.

This formula, together with some algebraic manipulations, implies that for simple $\mathbb{P}^r$ flops the quantum corrections attached to the extremal ray exactly remedy the defect caused by the classical product for any $r \in \mathbb{N}$ and the big quantum products restricted to exceptional curve classes are invariant under simple ordinary flops. Note that there are Novikov variables q involved in these transformations (c.f. Remark 3.3), and

$$\mathcal{F}(q^\beta) = q^{\mathcal{F}\beta}.$$

The proof has two ingredients: Localization and the divisor relations. Localization has been widely used in calculating Gromov–Witten invariants. For genus zero one-pointed descendent invariants twisted by a direct sum of negative line bundles, this was carried out in [16] and [7] in the context of the study of *mirror symmetry*. The divisor relations studied in [13] gives a *reconstruction theorem*, which allows us to go from one-point invariants to multiple-point ones.

To achieve the invariance of big quantum product, non-extremal curve classes need to be analyzed. The main purpose of §4 is to reduce the case of general X to the local case. Briefly, the degeneration formula expresses $\langle\alpha\rangle^X$ in terms of relative invariants $\langle\alpha_1\rangle^{(Y,E)}$ and $\langle\alpha_2\rangle^{(\tilde{E},E)}$, where $Y \to X$ is the blow-up of X over Z and $\tilde{E} = \mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O})$. Similarly for X', one has $Y', \tilde{E}', E'$. By definition of ordinary flops, $Y = Y'$ and $E = E'$. It is possible to match all output on the part of (Y,E) from X and X'. Thus, the problem is transformed to one for the relative cases of $(\tilde{E},E)$ and $(\tilde{E}',E)$. Following ideas in the work of D. Maulik and R. Pandharipande [20], a further reduction from relative invariants to absolute invariants is made. The problem is thus reduced to

$$X = \tilde{E} = \mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(-1)^{\oplus(r+1)} \oplus \mathcal{O}),$$

which is a semi-Fano projective bundle.

Remark 0.7. *For simple flops, we may and will consider only cohomology insertions of real even degrees throughout all our discussions on GW invariants. This is allowed since $\tilde{E}$ has only algebraic classes and any real odd degree insertion must go to the Y side after degeneration.*

The proof of the local case is carried out in §5 by exploring the compatibility of functional equations of n-point functions under the reconstruction procedure of genus zero invariants. It is easy to see that the Mori cone

$$NE(X) = \mathbb{Z}_+\ell \oplus \mathbb{Z}_+\gamma$$

with ℓ the line class in Z and γ the fiber line class of $X = \tilde{E} \to Z$. The proof is based on an induction on d_2 and n with degree $\beta = d_1\ell + d_2\gamma$. The case $d_2 = 0$ is handled by Theorem 0.6. For $d_2 > 0$, the starting case, namely the one-point invariant, is again based on localization technique on semi-Fano toric manifolds [7] and [17].

Theorem 0.8 (Functional equations for local models). *Consider an n-point function on $X = \mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(-1)^{\oplus(r+1)} \oplus \mathcal{O})$,*

$$\langle\alpha\rangle = \sum_{\beta\in NE(X)} \langle\alpha_1, \ldots, \alpha_n\rangle_\beta\, q^\beta$$

where α_i lies in the span of cohomology classes in X and descendents of (push-forward of) cohomology classes in E. For $\beta = d_1\ell + d_2\gamma$, the summands are non-trivial only for a fixed d_2. If $d_2 \neq 0$ then

$$\mathcal{F}\langle\alpha\rangle^X \cong \langle\mathcal{F}\alpha\rangle^{X'}.$$

(Here $\cong$ stands for equality up to analytic continuations.) Combining all the previous results Theorem 0.2 is proved.

REMARK 0.9. *Concerning ingredient (1), it is very important to understand the closure of ordinary flops. To the authors' knowledge, no serious attempt was made toward a higher dimensional version of (1) except some much weaker topological results [27]. Even in dimension three, the only known proof of (1) relies on the minimal model theory and classifications of terminal singularities. It is desirable to have a direct proof in the symplectic category. Such a proof should shed important light toward the higher dimensional cases. Our main theorem applies to K-equivalent maps that are composite of simple ordinary flops and their limits.*

As an application of the construction of ordinary flops in §1, we discuss (twisted) Mukai flops in §6. Some new understanding of the local geometry of Mukai flops is presented and this leads to a proof of Theorem 0.3. Theorem 0.3 can also be interpreted as a generalization of a local version of Huybrechts' results on hyper-Kähler manifolds [9], with the flexibility of allowing the base S to be any smooth variety. As in the hyper-Kähler case, it also implies that the correspondence induced by the fiber product

$$[X \times_{\bar{X}} X'] = [\bar{\Gamma}_f] + [Z \times_S Z'] \in A^*(X \times X')$$

is the one which gives an isomorphism of Chow motives.

Besides dimension three [15] and the hyper-Kähler case [9], our results provide the first known series of examples in all high dimensions which support Conjecture 0.4.

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1. Ordinary flops

1.1. Ordinary $\mathbb{P}^r$ flops.

Let $\psi : X \to \bar{X}$ be a flopping contraction as defined in §0.1. Our first task is to show that the corresponding algebraic ordinary flop $X \dashrightarrow X'$ exists. The construction of the desired flop is rather straightforward. First blow up X along Z to get $\phi : Y \to X$. The exceptional divisor E is a $\mathbb{P}^r \times \mathbb{P}^r$-bundle over S. The key point is that one may blow down E along another fiber direction $\phi' : Y \to X'$, with exceptional loci $\bar{\psi}' : Z' = \mathbb{P}_S(F') \to S$ for F' another rank $r+1$ vector bundle over S and also $N_{Z'/X'}|_{\bar{\psi}'-\text{fiber}} \cong \mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)}$. We start with the following elementary lemma.

Lemma 1.1. *Let $p : Z = \mathbb{P}_S(F) \to S$ be a projective bundle over S and $V \to Z$ a vector bundle such that $V|_{p^{-1}(s)}$ is trivial for every $s \in S$. Then $V \cong p^*F'$ for some vector bundle F' over S.*

Proof. Recall that $H^i(\mathbb{P}^r, \mathcal{O})$ is zero for $i \neq 0$ and $H^0(\mathbb{P}^r, \mathcal{O}) \cong \mathbb{C}$. By the theorem on Cohomology and Base Change we conclude immediately that $p_*\mathcal{O}(V)$ is locally free over S of the same rank as V. The natural map between locally free sheaves $p^*p_*\mathcal{O}(V) \to \mathcal{O}(V)$ induces isomorphisms over each fiber and hence by the Nakayama Lemma it is indeed an isomorphism. The desired F' is simply the vector bundle associated to $p_*\mathcal{O}(V)$. $\square$

Now apply the lemma to $V = \mathcal{O}_{\mathbb{P}_S(F)}(1) \otimes N_{Z/X}$, and we conclude that

$$N_{Z/X} \cong \mathcal{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\psi}^*F'.$$

Therefore, on the blow-up $\phi : Y = \mathrm{Bl}_Z X \to X$,

$$N_{E/Y} = \mathcal{O}_{\mathbb{P}_Z(N_{Z/X})}(-1).$$

¿From the Euler sequence which defines the universal sub-line bundle we see easily that $\mathcal{O}_{\mathbb{P}_Z(L\otimes F)}(-1) = \bar{\phi}^*L \otimes \mathcal{O}_{\mathbb{P}_Z(F)}(-1)$ for any line bundle L over Z. Since the projectivization functor commutes with pull-backs, we have

$$E = \mathbb{P}_Z(N_{Z/X}) \cong \mathbb{P}_Z(\bar{\psi}^*F') = \bar{\psi}^*\mathbb{P}_S(F') = \mathbb{P}_S(F) \times_S \mathbb{P}_S(F').$$

For future reference we denote the projection map $Z' := \mathbb{P}_S(F') \to S$ by $\bar{\psi}'$ and $E \to Z'$ by $\bar{\phi}'$. The various sets and maps are summarized in the following commutative diagram.

$$\begin{tikzcd}
 & E = \mathbb{P}_S(F) \times_S \mathbb{P}_S(F') \subset Y \arrow[dl, "\bar{\phi}"'] \arrow[dr, "\bar{\phi}'"] & \\
Z = \mathbb{P}_S(F) \subset X \arrow[dr, "\bar{\psi}"'] & & Z' = \mathbb{P}_S(F') \arrow[dl, "\bar{\psi}'"] \\
 & S \subset \bar{X} &
\end{tikzcd}$$

with normal bundle of E in Y being

$$\begin{aligned} N_{E/Y} &= \mathcal{O}_{\mathbb{P}_Z(N_{Z/X})}(-1) = \mathcal{O}_{\mathbb{P}_Z(\mathcal{O}_Z(-1)\otimes\bar{\psi}^*F')}(-1) \\ &= \bar{\phi}^*\mathcal{O}_{\mathbb{P}_S(F)}(-1) \otimes \mathcal{O}_{\mathbb{P}_Z(\bar{\psi}^*F')}(-1) \\ &= \bar{\phi}^*\mathcal{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\phi}'^*\mathcal{O}_{\mathbb{P}_S(F')}(-1). \end{aligned}$$

REMARK 1.2. *Notice that the bundles F and F' are uniquely determined up to a twisting by a line bundle. Namely, the pair (F, F') is equivalent to $(F \otimes L, F' \otimes L^*)$ for any line bundle L on S.*

The next step is to show that there is a blow-down map $\phi' : Y \to X'$ which contracts the left ruling of E and restricts to the projection map $\bar{\phi}' : E \to Z'$. The existence of the contraction $\psi : X \to \bar{X}$ is essential here. Let us denote a line in the left ruling by C_Y such that $\phi(C_Y) = C$.

PROPOSITION 1.3. *Ordinary $\mathbb{P}^r$ flops exist.*

Proof. Firstly, we will show that C_Y is K_Y-negative. From the exact sequence $0 \to T_C \to T_X|_C \to N_{C/X} \to 0$ and $N_{C/X} \cong \mathcal{O}_C(1)^{\oplus(r-1)} \oplus \mathcal{O}_C(-1)^{\oplus(r+1)} \oplus \mathcal{O}_C^{\dim S}$, we find that

$$(K_X.C) = 2g(C) - 2 - ((r-1) - (r+1)) = 0.$$

Together with $K_Y = \phi^*K_X + rE$, we get

$$(K_Y.C_Y) = (K_X.C) + r(E.C_Y) = -r < 0.$$

Next we will show C_Y is extremal, i.e. it has supporting (big and nef) divisors. Let H be a very ample divisor on X and L a supporting divisor for C (e.g. take $L = \phi^*\bar{H}$ for an ample divisor $\bar{H}$ on $\bar{X}$). Let $c = (H.C)$, then $\phi^*H + cE$ has type $(0, -c)$ on each $\mathbb{P}^r \times \mathbb{P}^r$ fiber of E. The divisor

$$k\phi^*L - (\phi^*H + cE)$$

is clearly big and nef for large k and vanishes precisely on the class $[C_Y]$. Thus C_Y is a K_Y-negative extremal ray and the contraction morphism $\phi' : Y \to X'$ fits into

$$\begin{tikzcd} Y \arrow[rr, "\phi'"] \arrow[dr, "\psi\circ\phi"'] & & X' \arrow[dl, "\psi'"] \\ & \bar{X} & \end{tikzcd}$$

by the cone theorem on $Y \to \bar{X}$ (c.f. [11]). $X \dashrightarrow X'$ is then the desired flop. $\square$

REMARK 1.4. *Notice that* $(K_X.C) = 0$, $(K_{X'}.C') = 0$ *(C' is a line in the fiber of $Z' \to S$) and* $\phi^* K_X = \phi'^* K_{X'}$ *(K-equivalence).*

It is clear from the proof that for the existence of ϕ' one needs only the (weaker) assumption that C is extremal instead of the existence of the contraction $\psi : X \to \bar{X}$. However, since $(K_X.C) = 0$ these two are indeed equivalent by the cone theorem.

1.2. Local models

In general, without assuming the existence of ψ, (i) and (ii) are not sufficient to construct ϕ' in the projective category. This is well known already in the case of Atiyah flop ($r = 1$ and $S = \{\mathrm{pt}\}$). In the analytic category results of Cornalba [4] do imply the contractibility of ψ, ϕ' and ψ' hence lead to the existence of analytic ordinary $\mathbb{P}^r$ flops under (i) and (ii). The situation is particularly simple in the case of *local models* which we now describe.

Consider a complex manifold S and two holomorphic vector bundles $F \to S$ and $F' \to S$. Let $\bar{\psi} : Z := \mathbb{P}_S(F) \to S$ and $\bar{\psi}' : Z' := \mathbb{P}_S(F') \to S$ be the induced morphisms and let $E = \mathbb{P}_S(F) \times_S \mathbb{P}_S(F')$ with two projections $\bar{\phi} : E \to Z$ and $\bar{\phi}' : E \to Z'$. Let Y be the total space of $N := \bar{\phi}^* \mathcal{O}_Z(-1) \otimes \bar{\phi}'^* \mathcal{O}_{Z'}(-1)$ with E the zero section. It is clear that $N_{E/Y} = N$. There is a contraction diagram

$$
\begin{tikzcd}
 & E \arrow[r, hook, "j"] \arrow[dl, "\pi_1 = \bar{\phi}"'] \arrow[dr, "\bar{\phi}' = \pi_2"] & Y \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\
Z \arrow[r, hook, "i"] \arrow[dr, "\bar{\psi}"'] & X \arrow[dr, "\psi"] & Z' \arrow[r, hook, "i'"] \arrow[dl, "\bar{\psi}'"'] & X' \arrow[dl, "\psi'"] \\
 & S \arrow[r, hook, "j'"] & \bar{X} &
\end{tikzcd}
$$

in the analytic category, with X (resp. X') being the total space of $\mathcal{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\psi}^* F'$ (resp. $\mathcal{O}_{\mathbb{P}_S(F')}(-1) \otimes \bar{\psi}'^* F$).

First of all, the discussion in §1.1 implies that ϕ and ϕ' are simply the blow-up maps along Z and Z' respectively. For ψ and ψ', when S reduces to a point the existence of contraction morphism $g : (Y, E) \to (\bar{X}, \mathrm{pt})$ is a classical result of Grauert since $N_{E/Y}$ is a negative line bundle. From the universal property the induced maps ψ and ψ' are then analytic. For S a small Stein open set, $g : (Y, E) \to (\bar{X}, S)$, as well as ψ and ψ', also exists since the whole picture is a trivial product with S. The general case follows from patching the local data over an open cover of S. In summary the local analytic model of an ordinary $\mathbb{P}^r$ flop is a locally trivial family (over S) of simple ordinary $\mathbb{P}^r$ flops.

It is convenient to consider compactified local models $\tilde{X}$, $\tilde{Y}$ etc. by adding the common infinity divisor $E_\infty \cong E$ to X, Y etc. respectively. Denote by

$$p : \tilde{X} = \mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O}_Z) \to Z.$$

Proposition 1.5. *If S is projective, for any bundles F, F' of rank $r+1$ the compactified local models of $\mathbb{P}^r$ flops exist in the projective category.*

Proof. $\tilde{X}$ is clearly projective and E_∞ is p-ample. By Remark 1.4 and Proposition 1.3 we only need to construct a supporting divisor L for the fiber line of $\bar{\psi} : Z \to S$. Let H be ample in S then $\bar{\psi}^* H$ is a supporting divisor for the fiber line in Z. Hence we may take $L := p^* \bar{\psi}^* H + E_\infty$. □

The projective local models will be used extensively in §4–§6.

2. Correspondences and motives

2.1. Grothendieck's category of Chow motives

General references of Chow motives can be found in [19] and [6]. Let $\mathcal{M}$ be the category of Chow motives (over $\mathbb{C}$). For each smooth variety X, one associates an object $\hat{X}$ in $\mathcal{M}$. The morphisms are given by correspondences

$$\mathrm{Hom}_{\mathcal{M}}(\hat{X}_1, \hat{X}_2) = A^*(X_1 \times X_2).$$

For $U \in A^*(X_1 \times X_2)$, $V \in A^*(X_2 \times X_3)$, let $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ be the projection maps. The composition law is given by

$$V \circ U = p_{13*}(p_{12}^* U . p_{23}^* V).$$

A correspondence U has associated maps on Chow groups:

$$U : A^*(X_1) \to A^*(X_2); \quad a \mapsto p_{2*}(U . p_1^* a)$$

as well as induced maps on T-valued points $\mathrm{Hom}(\hat{T}, \hat{X}_i)$:

$$U_T : A^*(T \times X_1) \xrightarrow{U\circ} A^*(T \times X_2).$$

Then we have *Manin's identity principle*: Let $U, V \in \mathrm{Hom}(\hat{X}, \hat{X}')$. Then $U = V$ if and only if $U_T = V_T$ for all T. (Since $U = U_X(\Delta_X) = V_X(\Delta_X) = V$.)

Theorem 2.1. *For an ordinary $\mathbb{P}^r$ flop $f : X \dashrightarrow X'$, the graph closure $\mathcal{F} := [\bar{\Gamma}_f]$ induces $\hat{X} \cong \hat{X}'$ via $\mathcal{F}^* \circ \mathcal{F} = \Delta_X$ and $\mathcal{F} \circ \mathcal{F}^* = \Delta_{X'}$.*

Proof. For any T, $\mathrm{id}_T \times f : T \times X \dashrightarrow T \times X'$ is also an ordinary $\mathbb{P}^r$ flop. Hence to prove that $\mathcal{F}^* \circ \mathcal{F} = \Delta_X$, by the identity principle, we only need to show that $\mathcal{F}^* \mathcal{F} = \mathrm{id}$ on $A^*(X)$ for any ordinary $\mathbb{P}^r$ flop. From the definition of pull-back,

$$\mathcal{F} W = p'_*(\bar{\Gamma}_f . p^* W) = \phi'_* \phi^* W.$$

We also have the formulae for pull-back from the intersection theory (c.f. [6], Theorem 6.7, Blow-up formula):

$$\phi^* W = \tilde{W} + j_*\big(c(\mathcal{E}) . \bar{\phi}^* s(W \cap Z, W)\big)_{\dim W}$$

where $\tilde{W}$ is the proper transform of W in Y and $\mathcal{E}$ is the excess normal bundle defined by

$$0 \to N_{E/Y} \to \phi^* N_{Z/X} \to \mathcal{E} \to 0 \tag{2.1.1}$$

and $s(W \cap Z, W)$ is the relative Segre class. The key observation is that the error term is lying over $W \cap Z$.

Let $W \in A_k(X)$. By Chow's moving lemma we may assume that W intersects Z transversally, so

$$\ell := \dim W \cap Z = k + (r+s) - (r+r+s+1) = k - r - 1.$$

Since $\dim \phi^{-1}(W \cap Z) = \ell + r = k - 1 < k$, the error term in the pull-back formula must be zero and we get $\phi^* W = \tilde{W}$. Hence $\mathcal{F}W = W'$, the proper transform of W in X'. Notice that W' is almost never transversal to Z'.

Let B be an irreducible component of $W \cap Z$ and $\bar{B} = \bar{\psi}(B) \subset S$ with dimension $\ell_B \le \ell$. Notice that $W' \cap Z'$ has irreducible components $\{B' := \bar{\psi}'^{-1}(\bar{B})\}_{B'}$ (different B with the same $\bar{B}$ will give rise to the same B').

Let $\phi'^* W' = \tilde{W} + \sum E_{B'}$, where $E_{B'}$ varies over irreducible components lying over B', hence $E_{B'} \subset \bar{\phi}'^{-1}\bar{\psi}'^{-1}(\bar{B})$, a $\mathbb{P}^r \times \mathbb{P}^r$ bundle over $\bar{B}$. For the generic point $s \in \psi(\phi(E_{B'})) \subset \bar{B}$, we thus have

$$\dim E_{B',s} \ge k - \ell_B = r + 1 + (\ell - \ell_B) > r.$$

In particular, $E_{B',s}$ contains positive dimensional fibers of ϕ (as well as ϕ'). Hence $\phi_*(E_{B'}) = 0$ and $\mathcal{F}^*\mathcal{F}W = W$.

By the same argument we have also that $\mathcal{F} \circ \mathcal{F}^* = \Delta_{X'}$, thus the proof is completed. □

Remark 2.2. *For a general ground field k, if the flop diagram under consideration is defined over k then the theorem works for motives over k.*

Corollary 2.3. *Let $f : X \dashrightarrow X'$ be a $\mathbb{P}^r$ flop. If $\dim \alpha_1 + \dim \alpha_2 = \dim X$, then*

$$(\mathcal{F}\alpha_1.\mathcal{F}\alpha_2) = (\alpha_1.\alpha_2).$$

That is, $\mathcal{F}$ is an isometry with respect to $(-.-)$.

Proof. We may assume that α_1, α_2 are transversal to Z. Then

$$\begin{aligned}(\alpha_1.\alpha_2) &= (\phi^*\alpha_1.\phi^*\alpha_2) = ((\phi'^*\mathcal{F}\alpha_1 - \xi).\phi^*\alpha_2) \\ &= ((\phi'^*\mathcal{F}\alpha_1).\phi^*\alpha_2) = (\mathcal{F}\alpha_1.(\phi'_*\phi^*\alpha_2)) = (\mathcal{F}\alpha_1.\mathcal{F}\alpha_2).\end{aligned}$$

Here we use the fact proved in the above theorem that ξ has positive fiber dimension in the ϕ direction. □

Thus for ordinary flops, $\mathcal{F}^{-1} = \mathcal{F}^*$ both in the sense of correspondences and Poincaré pairing.

REMARK 2.4. *It is an easy fact that if $X =_K X'$ then X and X' are isomorphic in codimension one and in particular the graph closure gives canonical isomorphisms $\mathcal{F}$ on $A^1(X) \cong A^1(X')$ and $A_1(X) \cong A_1(X')$ respectively. In this more general setting, the above proof still implies that the Poincaré pairing on $A^1 \times A_1$ (and $H^2 \times H_2$) is preserved under $\mathcal{F}$.*

2.2. Triple product for simple flops

Let $f : X \dashrightarrow X'$ be a simple $\mathbb{P}^r$ flop with S being a point. Let h be the hyperplane class of $Z = \mathbb{P}^r$ and h' be the hyperplane class of Z'. Let also $x = \bar{\phi}^* h = [h \times \mathbb{P}^r]$, $y = \bar{\phi}'^* h' = [\mathbb{P}^r \times h']$ in $E = \mathbb{P}^r \times \mathbb{P}^r$.

LEMMA 2.5. *For classes inside Z, we have*

$$\phi^*[h^l] = j_*(x^l y^r - x^{l+1} y^{r-1} + \cdots + (-1)^{r-l} x^r y^l).$$

Hence by symmetry we get $\mathcal{F}[h^l] = (-1)^{r-l}[h'^l]$. In particular, $\mathcal{F}[C] = -[C']$.

Proof. Recall that

$$N_{E/Y} = \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^r}(-1,-1) := \bar{\phi}^* \mathcal{O}_{\mathbb{P}^r}(-1) \otimes \bar{\phi}'^* \mathcal{O}_{\mathbb{P}^r}(-1)$$

and $N_{Z/X} = \mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)}$. ¿From (2.1.1),

$$c(\mathcal{E}) = (1-x)^{r+1}(1-x-y)^{-1}.$$

Taking degree r terms from both sides, we have

$$\begin{aligned}
c_r(\mathcal{E}) &= [(1-x)^{r+1}(1-(x+y))^{-1}]_{(r)} \\
&= (x+y)^r - C_1^{r+1}(x+y)^{r-1}x + \cdots + (-1)^r C_r^{r+1} x^r \\
&= (x+y)^{-1}((x+y)-x)^{r+1} - (-1)^{r+1}x^{r+1}) \\
&= (y^{r+1} - (-1)^{r+1}x^{r+1})/(y+x) \\
&= y^r - y^{r-1}x + y^{r-2}x^2 - \cdots + (-1)^r x^r.
\end{aligned}$$

The basic pull-back formula ([6], Proposition 6.7) then implies that

$$\phi^*[h^l] = j_*(c_r(\mathcal{E}).\bar{\phi}^*[h^l]) = j_*(c_r(\mathcal{E}).x^l) = j_* \sum\nolimits_{t=0}^{r} (-1)^t y^{r-t} x^{t+l}.$$

If $t + l \geq r + 1$ then $y^{r-t}x^{t+l} = 0$. The result follows. □

LEMMA 2.6. *For a class $\alpha \in H^{2l}(X)$ with $l \leq r$, let $\alpha' = \mathcal{F}\alpha$ in X'. Then*

$$\phi'^*\alpha' = \phi^*\alpha + (\alpha.h^{r-l})\, j_* \frac{x^l - (-y)^l}{x+y}.$$

Proof. Since the difference $\phi'^*\alpha' - \phi^*\alpha$ has support in E, we may write

$$\phi'^*\alpha' = \phi^*\alpha + j_*(a_1 x^{l-1} + \cdots + a_k x^{l-k} y^{k-1} + \cdots + a_l y^{l-1}).$$

By intersecting this equation with $x^{r-l}y^r$ in X and noticing that $E \sim -(x+y)$ on E, we get by the projection formula

$$0 = \phi^*\alpha.x^{r-l}y^r - a_1 x^{l-1}(x+y)x^{r-l}y^r = (\alpha.h^{r-l}) - a_1.$$

Similarly by intersecting with $x^{r-l+1}y^{r-1}$ we get

$$0 = -a_1 x^{l-1}(x+y)x^{r-l+1}y^{r-1} - a_2 x^{l-2}(x+y)x^{r-l+1}y^{r-1} = -a_1 - a_2.$$

Continuing in this way by intersecting with $x^p y^q$ with $p+q = 2r-l$ we get $a_k = (-1)^{k-1}(\alpha.h^{r-l})$ for all $k = 1, \ldots, l$. This proves the lemma. $\square$

These formulae allow us to compare the triple products of classes in X and X':

PROPOSITION 2.7. *For a simple $\mathbb{P}^r$-flop $f : X \dashrightarrow X'$, let $\alpha_i \in H^{2l_i}(X)$, with $l_i \le r$, $l_1 + l_2 + l_3 = \dim X = 2r+1$. Then*

$$(\mathcal{F}\alpha_1.\mathcal{F}\alpha_2.\mathcal{F}\alpha_3) = (\alpha_1.\alpha_2.\alpha_3) + (-1)^r(\alpha_1.h^{r-l_1})(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3}).$$

Proof. The proof consists of straightforward computations.

$$\begin{aligned}
(\mathcal{F}\alpha_1.\mathcal{F}\alpha_2.\mathcal{F}\alpha_3) &= (\phi'^*\mathcal{F}\alpha_1.\phi'^*\mathcal{F}\alpha_2.\phi'^*\mathcal{F}\alpha_3)\\
&= \left(\phi^*\alpha_1 + (\alpha_1.h^{r-l_1})j_*\frac{x^{l_1} - (-y)^{l_1}}{x+y}\right)\left(\phi^*\alpha_2 + (\alpha_2.h^{r-l_2})j_*\frac{x^{l_2} - (-y)^{l_2}}{x+y}\right)\\
&\qquad \times \left(\phi^*\alpha_3 + (\alpha_3.h^{r-l_3})j_*\frac{x^{l_3} - (-y)^{l_3}}{x+y}\right).
\end{aligned}$$

Among the resulting eight terms, the first term is clearly equal to $\alpha_1.\alpha_2.\alpha_3$.

For those three terms with two pull-backs like $\phi^*\alpha_1.\phi^*\alpha_2$, the intersection values are zero since the remaining part necessarily contains the ϕ fiber (from the formula the power in y is at most $l_3 - 1$).

The term with $\phi^*\alpha_1$ and two exceptional parts contributes

$$\begin{aligned}
&\phi^*\alpha_1.j_*\frac{x^{l_2} - (-y)^{l_2}}{x+y}.j_*\frac{x^{l_3} - (-y)^{l_3}}{x+y}\\
&= -\phi^*\alpha_1.j_*\big((x^{l_2} - (-y)^{l_2})(x^{l_3-1} + x^{l_3-2}(-y) + \cdots + (-y)^{l_3-1})\big)
\end{aligned}$$

times $(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3})$. The terms with non-trivial contribution must contain y^r, hence there is only one such term, namely (notice that $l_1 + l_2 + l_3 = 2r+1$)

$$-(-y)^{l_2} \times x^{l_3-1-(r-l_2)}(-y)^{r-l_2} = -(-1)^r x^{r-l_1} y^r$$

and the contribution is $(-1)^r(\alpha_1.h^{r-l_1})(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3})$. There are three such terms.

It remains to consider the term of triple product of three exceptional parts. It is $(\alpha_1.h^{r-l_1})(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3})$ times

$$(x^{l_1}-(-y)^{l_1})(x^{l_2}-(-y)^{l_2})(x^{l_3-1}+x^{l_3-2}(-y)+\cdots+(-y)^{l_3-1}).$$

The terms with non-trivial values are precisely multiples of x^ry^r. Since $l_1+l_2>r$, there are two such terms

$$-x^{l_1}(-y)^{l_2}\times x^{r-l_1}(-y)^{l_3-1-(r-l_1)}-x^{l_2}(-y)^{l_1}\times x^{r-l_2}(-y)^{l_3-1-(r-l_2)}$$

which give $-2(-1)^r$. Summing together we then finish the proof. $\square$

2.3. Motives and ordinary flips

Results in §1 and §2 extend straightforwardly to the case of ordinary flips. Before we move to quantum corrections for ordinary flops, we shall summarized here the classical aspects, especially the motivic aspects, of ordinary flips. The proofs are identical with the flop case and are thus omitted.

Consider $(\psi,\bar\psi):(X,Z)\to(\bar X,S)$ a log-extremal contraction as before. ψ is an ordinary (r,r') flipping contraction if

(i) $Z=\mathbb{P}_S(F)$ for some rank $r+1$ vector bundle F over S,

(ii) $N_{Z/X}|_{Z_s}\cong\mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus(r'+1)}$ for each $\bar\psi$-fiber Z_s, $s\in S$.

Then the (r,r') flip $f:X\dashrightarrow X'$ exists with explicit local model as in §1.2.

In terms of the K-partial order within a birational class, $X\le_K X'$ if and only if $r\le r'$. For f a (r,r') flip with $r\le r'$, the graph closure $\mathcal{F}=[\bar\Gamma_f]\in A^*(X\times X')$ identifies the Chow motive $\hat X$ of X as a sub-motive of $\hat X'$ which preserves also the Poincaré pairing on cohomology groups.

More precisely, a self correspondence $p\in A^*(X\times X)$ is a projector if $p^2=p$. There is a natural *pseduo-abelian extension* $\tilde{\mathcal{M}}$ of $\mathcal{M}$ to include all pairs (X,p) as its objects. (X,p) is regarded as the *image* of p. Moreover, $\hat X=(X,p)\oplus(X,1-p)$ in $\tilde{\mathcal{M}}$. With this notion, for an ordinary (r,r') flip $f:X\dashrightarrow X'$ with $r\le r'$, the graph closure $\mathcal{F}:=[\bar\Gamma_f]$ induces $\hat X\cong(X',p')$ via $\mathcal{F}^*\circ\mathcal{F}=\Delta_X$, where $p'=\mathcal{F}\circ\mathcal{F}^*$ is a projector.

Since every *geometric cohomology theory* (a graded ring functor H^* with Poincaré duality, Künneth formula and a cycle map $A^*\to H^*$ etc.) factors through $\tilde{\mathcal{M}}$, the result also holds on such a specialized theory.

For simple (r,r') flips (i.e. $S=\mathrm{pt}$) with $l\le\min\{r,r'\}$,

$$\phi^*[h^{r-l}]=j_*(x^{r-l}y^{r'}-x^{r-l+1}y^{r'-1}+\cdots+(-1)^lx^ry^{r'-l}).$$

In particular $\mathcal{F}[h^{r-l}] = (-1)^l [h'^{r'-l}]$. For $\alpha \in A^l(X)$ with $l \le \min\{r, r'\}$,

$$\phi'^* \mathcal{F}\alpha = \phi^*\alpha + (\alpha.h^{r-l})\, j_* \frac{x^l - (-y)^l}{x + y}.$$

Let $\alpha_i \in H^{2l_i}(X)$, $1 \le i \le 3$ with $l_i \le \min\{r, r'\}$, $l_1 + l_2 + l_3 = \dim X = r + r' + 1$. The defect of the triple product is again given by

$$(\mathcal{F}\alpha_1.\mathcal{F}\alpha_2.\mathcal{F}\alpha_3) = (\alpha_1.\alpha_2.\alpha_3) + (-1)^{r'}(\alpha_1.h^{r-l_1})(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3}).$$

3. Quantum corrections attached to extremal rays

Proposition 2.7 on triple products suggests that one needs to correct the product structure by some contributions from the extremal ray. In this section we show that for simple ordinary flops the quantum corrections attached to the extremal ray exactly remedy the defect of the ordinary product.

3.1. Quantum cohomology

We use [5] as our general reference on moduli spaces of stable maps, Gromov-Witten theory and quantum cohomology.

Let $\beta \in NE(X)$, the Mori cone of numerical classes of effective one cycles. Let $\overline{M}_{g,n}(X, \beta)$ be the moduli space of n-pointed stable maps $f : (C; x_1, \ldots, x_n) \to X$ from a nodal cure C with arithmetic genus $g(C) = g$ and with degree $[f(C)] = \beta$. Let $e_i : \overline{M}_{g,n}(X, \beta) \to X$ be the evaluation morphism $f \mapsto f(x_i)$. The Gromov-Witten invariant for classes $\alpha_i \in H^*(X)$, $1 \le i \le n$, is given by

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{g,n,\beta} := \int_{[\bar{M}_{g,n}(X,\beta)]^{virt}} e_1^*\alpha_1 \cdots e_n^*\alpha_n.$$

The genus zero three-point functions (as formal power series)

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle := \sum\nolimits_{\beta \in A_1(X)} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,\beta}\, q^\beta$$

together with the Poincaré pairing $(-, -)$ determine the *small* quantum product.

More precisely, let $T = \sum t_i T_i$ with $\{T_i\}$ a cohomology basis and t_i being formal variables. Let $\{T^i\}$ be the dual basis with $(T^i, T_j) = \delta_{ij}$. The (genus zero) pre-potential combines all n-point functions together:

$$\Phi(T) = \sum\nolimits_{n=0}^\infty \sum\nolimits_{\beta \in NE(X)} \frac{1}{n!} \langle T^n \rangle_\beta\, q^\beta,$$

where $\langle T^n \rangle_\beta = \langle T, \ldots, T \rangle_{0,n,\beta}$. The *big quantum product* is defined by

$$T_i *_t T_j = \sum\nolimits_k \Phi_{ijk} T^k$$

where

$$\Phi_{ijk} = \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k} = \sum\nolimits_{n=0}^{\infty} \sum\nolimits_{\beta \in NE(X)} \frac{1}{n!} \langle T_i, T_j, T_k, T^n \rangle_\beta \, q^\beta.$$

The *small quantum product* is defined to be the restriction of $*_t$ to $t = 0$ (the $n = 0$ part $\Phi_{ijk}(0)$).

In general, it is difficult to calculate $\Phi(T)$ and the big quantum ring directly. When X admits symmetries and $H^*(X)$ is *generated by divisors*, it is usually possible to use localization techniques to calculate one point invariants with gravitational descendents, or its generating function, the J-function, defined as follows.

$$(3.1.1)\quad \begin{aligned} J_X(q, z^{-1}) &:= \sum_{\beta \in NE(X)} q^\beta J_X(\beta, z^{-1}) \in H^*(X)[\![z^{-1}]\!][\![q]\!] \\ &:= \sum_{\beta \in NE(X)} q^\beta e_{1*}^X \left(\frac{1}{z(z-\psi)} \cap [\overline{M}_{0,1}(X,\beta)]^{virt} \right) \end{aligned}$$

Furthermore, the reconstruction theorem in [13] (also [2]) implies that J-function actually determines the entire generation function $\Phi(T)$.

3.2. Analytic continuation

Let $f : X \dashrightarrow X'$ be a simple $\mathbb{P}^r$ flop. Since X and X' have the same Poincaré pairing under $\mathcal{F}$, in order to compare their quantum products we only need to compare their n-point functions. For three-point functions, write

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = (\alpha_1.\alpha_2.\alpha_3) + \sum\nolimits_{d \in \mathbb{N}} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{d\ell} \, q^{d\ell} + \sum\nolimits_{\beta \notin \mathbb{Z}\ell} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_\beta \, q^\beta.$$

The difference $(\mathcal{F}\alpha_1.\mathcal{F}\alpha_2.\mathcal{F}\alpha_3) - (\alpha_1.\alpha_2.\alpha_3)$ is already determined in last section. The next step is to compute the middle term, namely quantum corrections coming from the extremal ray $\ell = [C]$. The third term will be discussed in later sections.

The virtual dimension of $\overline{M}_{g,n}(X, d\ell)$ is given by

$$(c_1(X).d\ell) + (2r+1)(1-g) + (3g-3) + n.$$

Since $(K_X.\ell) = 0$, for $g = 0$ we need only consider classes $\alpha_i \in A^{l_i}(X)$ with $\sum_{i=1}^n l_i = 2r + 1 + (n-3)$. For $n = 3$ this is $2r + 1 = \dim X$.

Theorem 3.1. *For all $\alpha_i \in H^{2l_i}(X)$ with $1 \le l_i \le r$, $\sum_{i=1}^n l_i = 2r + 1 + (n-3)$ and $d \in \mathbb{N}$,*

$$\begin{aligned} \langle \alpha_1, \ldots, \alpha_n \rangle_{0,n,d} &\equiv \int_{[\overline{M}_{0,n}(X,d\ell)]^{virt}} e_1^*\alpha_1 \cdots e_n^*\alpha_n \\ &= (-1)^{(d-1)(r+1)} N_{l_1,\ldots,l_n} d^{n-3} (\alpha_1.h^{r-l_1}) \cdots (\alpha_n.h^{r-l_n}). \end{aligned}$$

where $N_{l_1,\ldots,l_n}$ are recursively determined universal constants. $N_{l_1,\ldots,l_n}$ are independent of d and $N_{l_1,\ldots,l_n} = 1$ for $n = 2$ or 3. All other (primary) Gromov-Witten invariants with degree in $\mathbb{Z}\ell$ vanish.

Corollary 3.2. *Both the small and big quantum products restricted to exceptional curve classes are invariant under simple ordinary flops. In fact the three-point functions attached to the extremal ray exactly remedy the defect caused by the classical product.*

Proof. Since $(\mathcal{F}\alpha_i.h'^{(r-l_i)}) = (-1)^{l_i}(\mathcal{F}\alpha_i.\mathcal{F}h^{r-l_i}) = (-1)^{l_i}(\alpha_i.h^{r-l_i})$, for three point functions we get

$$\begin{aligned}\langle \mathcal{F}\alpha_1, \mathcal{F}\alpha_2, \mathcal{F}\alpha_3\rangle - \langle \alpha_1,\alpha_2,\alpha_3\rangle &= (-1)^r(\alpha_1.h^{r-l_1})(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3})\\ &+ (\alpha_1.h^{r-l_1})(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3})\left(\frac{(-1)^{2r+1}q^{\ell'}}{1-(-1)^{r+1}q^{\ell'}} - \frac{q^{\ell}}{1-(-1)^{r+1}q^{\ell}}\right).\end{aligned}$$

Under the correspondence $\mathcal{F}$, we shall identify $q^{\ell'}$ with $q^{-\ell}$. Plug in this into the last bracket we get 1 when r is odd and get -1 when r is even. In both cases the right hand side cancels out and then $\langle \mathcal{F}\alpha_1, \mathcal{F}\alpha_2, \mathcal{F}\alpha_3\rangle = \langle \alpha_1,\alpha_2,\alpha_3\rangle$. This proves the statement on small quantum product.

For general $n = 3+k$ point invariants with $k \geq 1$, we get

$$\begin{aligned}\langle \alpha_1,\ldots,\alpha_n\rangle &= N_{l_1,\ldots,l_n}(\alpha_1.h^{r-l_1})\cdots(\alpha_n.h^{r-l_n})\sum_{d=1}^{\infty}(-1)^{(d-1)(r+1)}d^k q^{d\ell}\\ &= N_{l_1,\ldots,l_n}(\alpha_1.h^{r-l_1})\cdots(\alpha_n.h^{r-l_n})\left(q^{\ell}\frac{d}{dq^{\ell}}\right)^k\frac{(-1)^{r+1}}{1-(-1)^{r+1}q^{\ell}}.\end{aligned}$$

Similarly, since $(-1)^{\sum l_i} = (-1)^{k+1}$, $\langle \mathcal{F}\alpha_1,\ldots,\mathcal{F}\alpha_n\rangle$ equals

$$(-1)^{k+1}N_{l_1,\ldots,l_n}(\alpha_1.h^{r-l_1})\cdots(\alpha_n.h^{r-l_n})\left(q^{\ell'}\frac{d}{dq^{\ell'}}\right)^k\frac{(-1)^{r+1}}{1-(-1)^{r+1}q^{\ell'}}.$$

Taking into account of

$$q^{-\ell}\frac{d}{dq^{-\ell}} = -q^{\ell}\frac{d}{dq^{\ell}} \quad\text{and}\quad \frac{1}{1-(-1)^{r+1}q^{-\ell}} = 1 - \frac{1}{1-(-1)^{r+1}q^{\ell}}$$

we get $\langle \mathcal{F}\alpha_1,\ldots,\mathcal{F}\alpha_n\rangle = \langle \alpha_1,\ldots,\alpha_n\rangle$ for all $k \geq 1$ ($n \geq 4$). The proof for the statement on big quantum product is thus completed. □

To put the result into perspective, we interpret the change of variable ℓ' by $-\ell$ in terms of analytic continuation over the extended complexified Kähler moduli space.

Without lose of generality we illustrate this by writing out the small quantum part. This is simply a word by word adoption of the treatment in the $r = 1$ case (cf. [28] §5.5, [21] §4).

The quantum cohomology is parameterized by the complexified Kähler class $\omega = B + iH$ with $q^\beta = \exp(2\pi i(\omega.\beta))$, where $B \in H^{1,1}_{\mathbb{R}}(X)$ and $H \in \mathcal{K}_X$, the Kähler cone of X. For a simple $\mathbb{P}^r$ flop $X \dashrightarrow X'$, $\mathcal{F}$ identifies $H^{1,1}$, A_1 and the Poincaré pairing $(-,-)$ on X and X'. Then $\langle \alpha_1, \alpha_2, \alpha_3\rangle^X$ restricted to $\mathbb{Z}\ell$ converges in the region

$$H^{1,1}_+ = \{\omega \,|\, (H.\ell) > 0\} \supset H^{1,1}_{\mathbb{R}} \times i\,\mathcal{K}_X$$

and equals

$$(\alpha_1.\alpha_2.\alpha_3) + (\alpha_1.h^{r-l_1})(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3})\frac{e^{2\pi i(\omega.\ell)}}{1-(-1)^{r+1}e^{2\pi i(\omega.\ell)}}.$$

This is a well-defined analytic function of ω on the whole $H^{1,1}$, which defines the analytic continuation of $\langle \alpha_1, \alpha_2, \alpha_3\rangle^X$ from $H^{1,1}_{\mathbb{R}} \times i\,\mathcal{K}_X$ to $H^{1,1}$.

Similarly, $\langle \mathcal{F}\alpha_1, \mathcal{F}\alpha_2, \mathcal{F}\alpha_3\rangle^{X'}$ restricted to $\mathbb{Z}\ell'$ converges in the region

$$\{\omega \,|\, (H.\ell') > 0\} = \{\omega \,|\, (H.\ell) < 0\} = H^{1,1}_- \supset H^{1,1}_{\mathbb{R}} \times i\,\mathcal{K}_{X'}$$

and equals

$$(\mathcal{F}\alpha_1.\mathcal{F}\alpha_2.\mathcal{F}\alpha_3) - (\alpha_1.h^{r-l_1})(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3})\frac{e^{-2\pi i(\omega.\ell)}}{1-(-1)^{r+1}e^{-2\pi i(\omega.\ell)}}$$

which is the analytic continuation of the previous one from $H^{1,1}_+$ to $H^{1,1}_-$.

We introduce the notation $A \cong B$ for the two series A and B when they can be analytically continued to each other.

Remark 3.3. *It was conjectured that the total series Φ^X_{ijk} converges for $B \in \mathcal{K}_X$, at least for B large enough, hence the large radius limit goes back to the classical cubic product. The Novikov variables $\{q^\beta\}_{\beta \in NE(X)}$ are introduced to avoid the convergence issue.*

Since $\mathcal{K}_X \cap \mathcal{K}_{X'} = \emptyset$ for non-isomorphic K-equivalent models, the collection of Kähler cones among them form a chamber structure. The conjectural canonical isomorphism

$$\mathcal{F} : H^*(X) \cong H^*(X')$$

assigns to each model X a coordinate system $H^*(X)$ of the fixed H^* and $\mathcal{F}$ serves as the (linear) transition function. The conjecture asserts that Φ^X_{ijk} can be analytically continued from $\mathcal{K}_X$ to $\mathcal{K}_{X'}$ and agrees with $\Phi^{X'}_{ijk}$. Equivalently, Φ_{ijk} is well-defined on $\mathcal{K}_X \cup \mathcal{K}_{X'}$ which verifies the functional equation

$$\mathcal{F}\Phi_{ijk}(\omega, T) \cong \Phi_{ijk}(\omega, \mathcal{F}T).$$

For simple ordinary flops, this is verified from §3 to §5 for each given cohomology insertions. The convergence has just been verified for extremal rays and will be verified for local models in §5.

3.3. One-point functions with descendents

In order to prove Theorem 3.1, we first reduce the problem to one for projective spaces. Let

$$U_d := R^1 ft_* e_{n+1}^* N$$

be the obstruction bundle, where $N = N_{Z/X}$ and ft is the forgetting morphism in

$$\begin{array}{ccc} \overline{M}_{0,n+1}(\mathbb{P}^r, d) & \xrightarrow{e_{n+1}} & \mathbb{P}^r \ . \\ \Big\downarrow{\scriptstyle ft} & & \\ \overline{M}_{0,n}(\mathbb{P}^r, d) & & \end{array}$$

It is well known (see e.g. [5]) that

$$[\overline{M}_{0,n}(X, d\ell)]^{virt} = e(U_d) \cap [\overline{M}_{0,n}(\mathbb{P}^r, d\ell)]. \tag{3.3.1}$$

Since U_d is functorial under ft^*, we use the same notation for all n.

As explained earlier, we will start with the calculation of the J-function (3.1.1). In our case

$$J_X(d\ell, z^{-1}) \equiv e_{1*}^{\mathbb{P}^r} \frac{e(U_d)}{z(z-\psi)}$$

has been calculated: Let $P_d := (-1)^{(d-1)(r+1)} \dfrac{1}{(h+dz)^{r+1}}$.

Lemma 3.4 ([16], also [7]).

$$J_X(d\ell, z^{-1}) = P_d.$$

Remark 3.5. *This calculation can be interpreted as quantum Lefschetz hyperplane theorem for concave bundles over $\mathbb{P}^r$. From this viewpoint, the "mirror transformation" from $J_X(d\ell, z^{-1})$ to P_d is not needed since the rank of the bundle $\mathcal{O}(-1)^{r+1}$ is greater than one. See e.g. [12].*

Corollary 3.6. *For $l + k = 2r - 1$, $1 \le l \le r$,*

$$\left\langle \tau_k h^l \right\rangle_d = \frac{(-1)^{d(r+1)+k}}{d^{k+2}} C_r^{k+1}$$

where $C_r^k = k!/r!(k-r)!$. The invariant is zero if $l+k \neq 2r-1$ by dimensional constraints.

Proof. We start with

$$A := \int_{\mathbb{P}^r} h^l . P_d = \sum\nolimits_{k \ge 0} \frac{1}{z^{k+2}} \left\langle \tau_k h^l \right\rangle_d .$$

By Lemma 3.4

$$A = (-1)^{(d-1)(r+1)} \int_{\mathbb{P}^r} \frac{h^l}{(h+dz)^{r+1}} = \int_{\mathbb{P}^r} \frac{h^l}{d^{r+1}z^{r+1}} \left(1 + \frac{h}{dz}\right)^{-(r+1)}.$$

The result follows from the Taylor expansion and the elementary fact that

$$C_{r-l}^{-(r+1)} = (-1)^{k+(r+1)} C_r^{k+1}.$$

□

3.4. Multiple-point functions via divisor relations

We recall the following rational equivalence in $A_*(\overline{M}_{0,n}(X,\beta)) \otimes \mathbb{Q}$ from [13], Corollary 1: For $L \in \operatorname{Pic}(X)$ and $i \neq j$,

$$\begin{aligned} &e_i^* L \cap [\overline{M}_{0,n}(X,\beta)]^{virt} \\ =&(e_j^* L + (\beta, L)\psi_j) \cap [\overline{M}_{0,n}(X,\beta)]^{virt} - \sum_{\beta_1+\beta_2=\beta} (\beta_1, L)[D_{i,\beta_1|j,\beta_2}]^{virt}, \end{aligned} \tag{3.4.1}$$

$$\psi_i + \psi_j = [D_{i|j}]^{virt}, \tag{3.4.2}$$

where $[D_{i,\beta_1|j,\beta_2}]^{virt} \in A_*(\overline{M}_{0,n}(X,\beta))$ is the push-forward of the virtual classes of the corresponding boundary divisor components

$$D_{i,\beta_1|j,\beta_2} = \sum_{i\in A, j \in B;\ A\coprod B = \{1,\ldots,n\}} D(A,B;\beta_1,\beta_2)$$

and

$$D_{i|j} = \sum_{\beta_1+\beta_2=\beta} D_{i,\beta_1|j,\beta_2}.$$

Here is a simple observation which will be repeatedly used in the sequel:

LEMMA 3.7 (Vanishing lemma). *Let $\mathbb{P}^r \subset X$ with $N_{\mathbb{P}^r/X} = \oplus_j \mathcal{O}(-m_j)$, $m_j \in \mathbb{N}$. Let ℓ be the line class in $\mathbb{P}^r$. Then for $\deg T > r$ and $d \neq 0$, $\langle \ldots, T\rangle_{d\ell} = 0$.*

Proof. Since $[\overline{M}_{0,n}(X,d\ell)]^{virt}$ equals $[\overline{M}_{0,n}(\mathbb{P}^r,d)]$ cut out by $e(U_d)$, the evaluation morphisms factor through $\mathbb{P}^r$. But then $e_n^*(T|_Z) = 0$. □

Here $\deg T := l$ if $T \in H^{2l}(X)$. As we had mentioned in the introduction, only real even degree classes will be relevant throughout our discussions.

PROPOSITION 3.8. *For $k_1 + k_2 + l_1 + l_2 = 2r$, $1 \leq l_i \leq r$,*

$$\left\langle \tau_{k_1} h^{l_1}, \tau_{k_2} h^{l_2} \right\rangle_d = \frac{(-1)^{d(r+1)+l_1+k_2+1}}{d^{k_1+k_2+1}} C_{r-l_1}^{2r-(l_1+l_2)},$$

and other descendent invariants vanish. In particular, the only non-trivial two-point function without descendents in degree $d\ell$ is given by

$$\langle h^r, h^r\rangle_d = (-1)^{(d-1)(r+1)}\frac{1}{d}.$$

Proof. We consider the invariant without descendents first. Since the virtual dimension is $2r$, only $\langle h^r, h^r\rangle_d$ survives. Using the above equivalence relations, we may decrease the power of e_1^*h one by one. In each step only the second term in the resulting three terms has nontrivial contribution. Indeed, for the first term any addition to the power of $e_2^*h^r$ leads to zero.

For the third boundary splitting terms, write $[\Delta(X)] = \sum_i T^i \otimes T_i$. For each i, since $\dim X = 2r+1$ one of T^i or T_i must have degree strictly bigger than r. If $\beta_1 = d_1\ell$, $\beta_2 = d_2\ell$ with $d_i \neq 0$ then one of the integral, hence the product, must vanish by the vanishing lemma.

This is what happens now. We apply the divisor relation to $i = 1$ and $j = 2$. Since $n = 2$, we find $n_1 = |A| = 1$, $n_2 = |B| = 1$ and in the splitting we have sum of product of two-point invariants. The degree in each side is non-zero since there is no constant genus zero stable map with two marked points. So the splitting terms vanish.

We apply the divisor relation repeatedly to compute

$$\langle h^r, h^r\rangle_d = d\langle h^{r-1}, \tau_1 h^r\rangle_d = \cdots = d^{r-1}\langle h, \tau_{r-1}h^r\rangle_d = d^r\langle \tau_{r-1}h^r\rangle$$

where the last equality is by the divisor axiom. Now we plug in Corollary 3.6 with $(k,l) = (r-1,r)$ and the statement follows.

For descendent invariants we proceed in the same manner. For simplicity we abuse the notation by denoting $\langle\cdots, \psi^s\alpha, \cdots\rangle_\beta = \langle\cdots, \tau_s\alpha, \cdots\rangle_\beta$. Let $s \geq 1$, $l + m + s = 2r$ and consider

$$\begin{aligned}\left\langle h^l, \psi^s h^m\right\rangle_d &= \left\langle h^{l-1}, \psi^s h^{m+1}\right\rangle_d + (h, d\ell)\left\langle h^{l-1}, \psi^{s+1} h^m\right\rangle_d \\ &= \left\langle h^{l-1}, (h + d\psi)\psi^s h^m\right\rangle_d = \cdots \\ &= \left\langle h, (h+d\psi)^{l-1}\psi^s h^m\right\rangle_d.\end{aligned}$$

Notice that the splitting terms are all zero as before. Now the divisor axiom of descendent invariants gives

$$d\left\langle (h+d\psi)^{l-1}\psi^s h^m\right\rangle_d + \left\langle (h+d\psi)^{l-1}\psi^{s-1}h^{m+1}\right\rangle_d,$$

which leads to the reduction formula:

$$\left\langle h^l, \psi^s h^m\right\rangle_d = \left\langle (h+d\psi)^l\psi^{s-1}h^m\right\rangle_d.$$

Notice that this equals the constant term in z in

$$\begin{aligned}
&\left\langle \sum_{k\geq 0} \frac{\psi^k}{z^k}(h+dz)^l z^{s-1} h^m \right\rangle_d \\
&= z^{s+1} e_{1*}\left(\frac{e(U_d)}{z(z-\psi)}.e_1^*\left((h+dz)^l.h^m\right)\right) \\
&= (-1)^{(d-1)(r+1)} z^{s+1}(h+dz)^{l-(r+1)}.h^m \\
&= (-1)^{(d-1)(r+1)} \frac{z^{r-m}}{d^{(r+1)-l}}\left(1+\frac{h}{dz}\right)^{l-(r+1)}.h^m,
\end{aligned}$$

which is

$$\frac{(-1)^{d(r+1)+r+1}}{d^{r+1-l+r-m}} C^{l-(r+1)}_{r-m} = \frac{(-1)^{d(r+1)+l+s+1}}{d^{s+1}} C^{2r-(l+m)}_{r-m}.$$

In general from $\psi_1 = -\psi_2 + [D_{1|2}]^{virt}$, we find

$$\left\langle \tau_{k_1} h^{l_1}, \tau_{k_2} h^{l_2} \right\rangle_d = -\left\langle \tau_{k_1-1} h^{l_1}, \tau_{k_2+1} h^{l_2} \right\rangle_d = \cdots = (-1)^{k_1}\left\langle h^{l_1}, \tau_{k_1+k_2} h^{l_2} \right\rangle_d$$

since the splitting terms all vanishes. The result follows. □

For $n \geq 3$, it is known that for any three different markings i, j and k, $\psi_j = [D_{ik|j}]^{virt}$. By plugging this into (3.4.1), we get

$$e_i^* L = e_j^* L + \sum_{\beta_1+\beta_2=\beta} ((\beta_2.L)[D_{ik,\beta_1|j,\beta_2}]^{virt} - (\beta_1.L)[D_{i,\beta_1|jk,\beta_2}]^{virt}).$$

In our special case this reads as

$$e_i^* h = e_j^* h + \sum_{d_1+d_2=d} (d_2[D_{ik,d_1|j,d_2}]^{virt} - d_1[D_{i,d_1|jk,d_2}]^{virt}).$$

Notice that now d_i is allowed to be zero.

LEMMA 3.9. *For $n \geq 3$,*

$$\begin{aligned}
&\langle h^{l_1+1}, h^{l_2}, h^{l_3}, \ldots\rangle_{n,d} \\
&= \langle h^{l_1}, h^{l_2+1}, h^{l_3}, \ldots\rangle_{n,d} + d\langle h^{l_1+l_3}, h^{l_2}, \ldots\rangle_{n-1,d} - d\langle h^{l_1}, h^{l_2+l_3}, \ldots\rangle_{n-1,d}.
\end{aligned}$$

Note that for $l_1 = 0$ this recovers the divisor axiom.

Proof. As in the previous theorem, the boundary terms with non-trivial degree must vanish. For degree zero, the only non-trivial invariants are three-point functions, hence we are left with

$$\begin{aligned}
&\langle h^{l_1+1}, h^{l_2}, h^{l_3}, \ldots\rangle_{n,d} \\
&= \langle h^{l_1}, h^{l_2+1}, h^{l_3}, \ldots\rangle_{n,d} \\
&\quad + \sum_i d\langle h^{l_1}, h^{l_3}, T_i\rangle_0 \langle T^i, h^{l_2}, \ldots\rangle_d - \sum_i d\langle T^i, h^{l_1}, \ldots\rangle_d \langle h^{l_2}, h^{l_3}, T_i\rangle_0.
\end{aligned}$$

For the first boundary sum, in the diagonal decomposition $[\Delta(X)] = \sum T_i \otimes T^i$ we may choose basis so that $h^{l_1+l_3}$ appear in $\{T^i\}$. Then the above degree zero invariants survive only in one term which is equal to 1. The same argument applies to the second sum too. So the above expression equals

$$\langle h^{l_1}, h^{l_2+1}, h^{l_3}, \ldots\rangle_d + d\langle h^{l_1+l_3}, h^{l_2}, \ldots\rangle_d - d\langle h^{l_1}, h^{l_2+l_3}, \ldots\rangle_d$$

as expected. □

In light of (3.3.1) and results above, Theorem 3.1 can be reformulated as the following equation

$$\langle h^{l_1}, h^{l_2}, \ldots, h^{l_n}\rangle_d = (-1)^{(d-1)(r+1)} N_{l_1,\ldots,l_n} d^{n-3}, \tag{3.4.3}$$

which we will now prove.

Proof. (of (3.4.3), or equivalently Theorem 3.1.)

We will prove the theorem by induction on $n \in \mathbb{N}$. The case $n \le 2$ are already proven before. We treat the case $n = 3$ first.

Consider $\langle h^{l_1}, h^{l_2}, h^{l_3}\rangle_d$ with $l_1 + l_2 + l_3 = 2r + 1$ and $l_1 \le l_2 \le l_3$. If $l_1 = 1$ then $l_2 = l_3 = r$ and so

$$\langle h, h^r, h^r\rangle_d = d\langle h^r, h^r\rangle_d = (-1)^{(d-1)(r+1)}.$$

If $l_1 \ge 2$, then $l_2 \le r - 1$ and

$$\langle h^{l_1}, h^{l_2}, h^{l_3}\rangle_d = \langle h^{l_1-1}, h^{l_2+1}, h^{l_3}\rangle_d + d\langle h^{l_1+l_3-1}, h^{l_2}\rangle_d - d\langle h^{l_1-1}, h^{l_2+l_3}\rangle_d.$$

But then both $l_1 + l_3 - 1$ and $l_2 + l_3$ are larger than $r + 1$ and the boundary terms vanish individually. By reordering l_2, l_3 if necessary, and repeating this procedure we are reduced to the case $l_1 = 1$ and proof for $n = 3$ is completed.

Suppose the theorem holds up to $n - 1$ (with $n \ge 4$). The above lemma and the induction hypothesis imply that

$$\begin{aligned}
&\langle h^{l_1}, h^{l_2}, h^{l_3}, \ldots\rangle_d \\
&= \langle h^{l_1-1}, h^{l_2+1}, h^{l_3}, \ldots\rangle_d + d\langle h^{l_1+l_3-1}, h^{l_2}, \ldots\rangle_d - d\langle h^{l_1-1}, h^{l_2+l_3}, \ldots\rangle_d \\
&= \langle h^{l_1-1}, h^{l_2+1}, h^{l_3}, \ldots\rangle_d + (N_{l_1+l_3-1,l_2,\ldots} - N_{l_1-1,l_2+l_3,\ldots})d^{n-3}.
\end{aligned}$$

By repeating this procedure, l_1 is decreased to one and we get

$$\langle h^{l_1}, h^{l_2}, \ldots, h^{l_n}\rangle_d = (-1)^{(d-1)(r+1)} N_{l_1,\ldots,l_n} d^{n-3},$$

where $N_{l_1,\ldots,l_n}$ is given by N_*'s in one lower level. The proof is complete. □

Similar methods apply to descendent invariants:

PROPOSITION 3.10. *The only three-point descendent invariants of extremal classes $d\ell$, up to permutations of insertions, are given by*

$$\left\langle h^{l_1}, h^{l_2}, \tau_{k_3} h^{l_3} \right\rangle_d = \frac{(-1)^{d(r+1)+l_3+1}}{d^{k_3}} C^{k_3+1}_{r-(l_1+l_2)},$$

where $l_1 + l_2 + l_3 + k_3 = 2r + 1$ and by convention $C^m_n = 0$ if $n < 0$.

More generally, an n-point descendent invariant $\langle \prod_{i=1}^n \tau_{k_i} h^{l_i} \rangle_d$ with $n \geq 3$ is non-zero only if there are at least two insertions being free of descendents, say $k_1 = k_2 = 0$. In such cases, there are universal constants $N_{k,l} \in \mathbb{Z}$ such that

$$\left\langle h^{l_1}, h^{l_2}, \tau_{k_3} h^{l_3}, \ldots, \tau_{k_n} h^{l_n} \right\rangle_d = N_{k,l}\, d^{n-3-\sum k_i}.$$

Proof. Let $n \geq 3$ and assume that $0 \leq k_1 \leq k_2 \leq \cdots \leq k_n$. If $k_2 \geq 1$ then use $\psi_2 = [D_{2|13}]^{virt}$ we get

$$\left\langle \tau_{k_1} h^{l_1}, \ldots, \tau_{k_n} h^{l_n} \right\rangle_d = \sum_{i;\, d_1+d_2=d} \left\langle \tau_{k_2-1} h^{l_2}, \cdots, T_i \right\rangle_{d_1} \left\langle T^i, \tau_{k_1} h^{l_1}, \tau_{k_3} h^{l_3}, \cdots \right\rangle_{d_2}.$$

We separate two cases. If the first factor is a two-point function then it is non-zero only if $T_i = h^j$ for some $j \leq r$. But then $\deg T^i > r$ and the right factor vanishes since it contains ψ classes. For other cases, both factors contain ψ classes hence the factor with $\deg T_i > r$ (or $\deg T^i > r$) must vanish.

For three-point invariants, from $\psi_3 = [D_{3|12}]^{virt}$ we get as before that

$$\begin{aligned} \left\langle h^{l_1}, h^{l_2}, \tau_{k_3} h^{l_3} \right\rangle_d &= \sum_{i;\, d_1+d_2=d} \left\langle \tau_{k_3-1} h^{l_3}, T_i \right\rangle_{d_1} \left\langle T^i, h^{l_1}, h^{l_2} \right\rangle_{d_2} \\ &= \left\langle \tau_{k_3-1} h^{l_3}, h^{l_1+l_2} \right\rangle_d \end{aligned}$$

and the formula follows from the two-point case.

Similarly, for $n \geq 4$, if $k_i \neq 0$ then from $\psi_i = [D_{i|12}]^{virt}$ we get

$$\left\langle h^{l_1}, h^{l_2}, \ldots, \tau_{k_i} h^{l_i}, \ldots \right\rangle_d = \left\langle \tau_{k_i-1} h^{l_i}, h^{l_1+l_2}, \ldots \right\rangle_d.$$

The result follows from an induction on n. □

4. Degeneration analysis

Our next task is to *compare* the genus zero Gromov–Witten invariants of X and X' for curve classes other than the flopped curve. Naively, one may wish to "decompose" the varieties into the neighborhoods of exceptional loci and their complements. As the latter's are obviously isomorphic, one is reduced to study the local case. The degeneration formula [15] [14] [10] provides a rigorous formulation of the above naive picture.

4.1. The degeneration formula

Our presentation of degeneration formula below mostly follows that of [14] and [18]. We have, however, chosen to use the "numerical form" rather than the "cycle form" in the exposition.

Given a relative pair (Y, E) with $E \hookrightarrow Y$ a smooth divisor, the relative Gromov–Witten invariants are defined in the following way. Let $\Gamma = (g, n, \beta, \rho, \mu)$ with $\mu = (\mu_1, \ldots, \mu_\rho) \in \mathbb{N}^\rho$ a partition of the intersection number $(\beta.E) = |\mu| := \sum_{i=1}^{\rho} \mu_i$. For $A \in H^*(Y)^{\otimes n}$ and $\varepsilon \in H^*(E)^{\otimes \rho}$, the relative invariant of stable maps with topological type Γ (i.e. with contact order μ_i in E at the i-th contact point) is

$$\langle A \mid \varepsilon, \mu \rangle_\Gamma^{(Y,E)} := \int_{[\overline{M}_\Gamma(Y,E)]^{virt}} e_Y^* A \cup e_E^* \varepsilon$$

where $e_Y : \overline{M}_\Gamma(Y, E) \to Y^n$, $e_E : \overline{M}_\Gamma(Y, E) \to E^\rho$ are evaluation maps on marked points and contact points respectively.

If $\Gamma = \coprod_\pi \Gamma^\pi$, the relative invariants (with disconnected domain curves)

$$\langle A \mid \varepsilon, \mu \rangle_\Gamma^{\bullet (Y,E)} := \prod\nolimits_\pi \langle A \mid \varepsilon, \mu \rangle_{\Gamma^\pi}^{(Y,E)}$$

are defined to be the product of the connected components.

We apply the degeneration formula to the following situation. Let X be a smooth variety and $Z \subset X$ be a smooth subvariety. Let $\Phi : W \to \mathfrak{X}$ be its *degeneration to the normal cone*, the blow-up of $X \times \mathbb{A}^1$ along $Z \times \{0\}$. Denote $t \in \mathbb{A}^1$ the deformation parameter. Then $W_t \cong X$ for all $t \neq 0$ and $W_0 = Y_1 \cup Y_2$ with

$$\phi = \Phi|_{Y_1} : Y_1 \to X$$

the blow-up along Z and

$$p = \Phi|_{Y_2} : Y_2 := \mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O}) \to Z \subset X$$

the projective completion of the normal bundle. $Y_1 \cap Y_2 =: E = \mathbb{P}_Z(N_{Z/X})$ is the ϕ-exceptional divisor which consists of "the infinity part" of the projective bundle $\mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O})$.

Since the family $W \to \mathbb{A}^1$ is a degeneration of a trivial family, all cohomology classes $\alpha \in H^*(X, \mathbb{Z})^{\oplus n}$ have global liftings and the restriction $\alpha(t)$ on W_t is defined for all t. Let $j_i : Y_i \hookrightarrow W_0$ be the inclusion maps for $i = 1, 2$. Let $\{e_i\}$ be a basis of $H^*(E)$ with $\{e^i\}$ its dual basis. $\{e_I\}$ forms a basis of $H^*(E^\rho)$ with dual basis $\{e^I\}$ where $|I| = \rho$, $e_I = e_{i_1} \otimes \cdots \otimes e_{i_\rho}$. The *degeneration formula* expresses the absolute invariants of X in terms of the relative invariants of the two smooth pairs (Y_1, E) and (Y_2, E):

$$\langle \alpha \rangle_{g,n,\beta}^X = \sum_I \sum_{\eta \in \Omega_\beta} C_\eta \left\langle j_1^* \alpha(0) \,\middle|\, e_I, \mu \right\rangle_{\Gamma_1}^{\bullet (Y_1,E)} \left\langle j_2^* \alpha(0) \,\middle|\, e^I, \mu \right\rangle_{\Gamma_2}^{\bullet (Y_2,E)}.$$

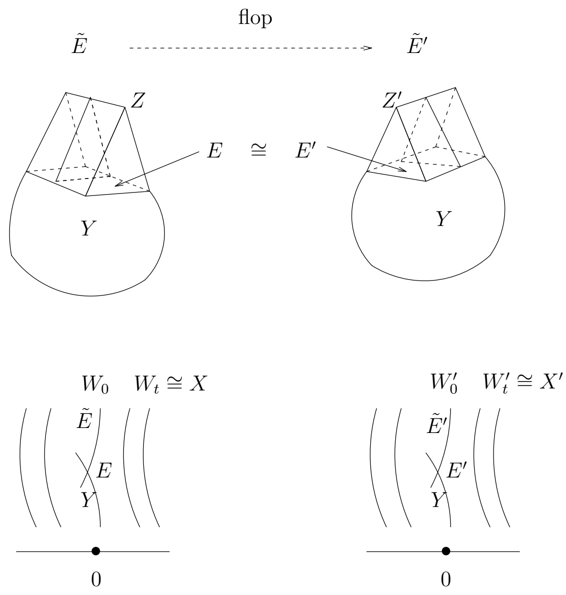

Figure 4.1.1: Degeneration to the normal cone for ordinary flops.

Here $\eta = (\Gamma_1, \Gamma_2, I_\rho)$ is an *admissible triple* which consists of (possibly disconnected) topological types

$$\Gamma_i = \coprod_{\pi=1}^{|\Gamma_i|} \Gamma_i^\pi$$

with the same partition μ of contact order under the identification I_ρ of contact points. The gluing $\Gamma_1 +_{I_\rho} \Gamma_2$ has type (g, n, β) and is connected. In particular, $\rho = 0$ if and only if that one of the Γ_i is empty. The total genus g_i, total number of marked points n_i and the total degree $\beta_i \in NE(Y_i)$ satisfy the splitting relations

$$\begin{aligned} g &= g_1 + g_2 + \rho + 1 - |\Gamma_1| - |\Gamma_2|, \\ n &= n_1 + n_2, \\ \beta &= \phi_* \beta_1 + p_* \beta_2. \end{aligned}$$

The constants $C_\eta = m(\mu)/|\mathrm{Aut}\,\eta|$, where $m(\mu) = \prod \mu_i$ and $\mathrm{Aut}\,\eta = \{\, \sigma \in S_\rho \mid \eta^\sigma = \eta \,\}$. (When a map is decomposed into two parts, an (extra) ordering to the contact points is assigned. The automorphism of the decomposed curves will also introduce an extra factor. These contribute to $\mathrm{Aut}\,\eta$.) We denote by Ω the set of equivalence classes of all admissible triples; by Ω_β and Ω_μ the subset with fixed degree β and fixed contact order μ respectively.

Given an ordinary flop $f : X \dashrightarrow X'$, we apply degeneration to the normal cone to both X and X'. Then $Y_1 \cong Y_1'$ and $E = E'$, by the definition of ordinary flops. The following notations will be used

$$Y := \mathrm{Bl}_Z X \cong Y_1 \cong Y_1', \quad \tilde{E} := \mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O}), \quad \tilde{E}' := \mathbb{P}_{Z'}(N_{Z'/X'} \oplus \mathcal{O}).$$

REMARK 4.1. *For simple $\mathbb{P}^r$ flops, $Y_2 \cong \mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(-1)^{\oplus(r+1)} \oplus \mathcal{O}) \cong Y_2'$. However the gluing maps of Y_1 and Y_2 along E for X and X' differ by a twist which interchanges the order of factors in $E = \mathbb{P}^r \times \mathbb{P}^r$. Thus $W_0 \ncong W_0'$ and it is necessary to study the details of the degenerations. In general, f induces an ordinary flop $\tilde{f} : Y_2 \dashrightarrow Y_2'$ of the same type which is the local model of f.*

4.2. Liftings of cohomology insertions

Next we discuss the presentation of $\alpha(0)$. Denote by $\iota_1 \equiv j : E \hookrightarrow Y_1 = Y$ and $\iota_2 : E \hookrightarrow Y_2 = \tilde{E}$ the natural inclusions. The class $\alpha(0)$ can be represented by $(j_1^*\alpha(0), j_2^*\alpha(0)) = (\alpha_1, \alpha_2)$ with $\alpha_i \in H^*(Y_i)$ such that

$$\iota_1^*\alpha_1 = \iota_2^*\alpha_2 \quad \text{and} \quad \phi_*\alpha_1 + p_*\alpha_2 = \alpha. \tag{4.2.1}$$

Such representatives are called *liftings* which are by no means unique. The flexibility on different choices will be useful.

One choice of the lifting is

$$\alpha_1 = \phi^*\alpha \quad \text{and} \quad \alpha_2 = p^*(\alpha|_Z), \tag{4.2.2}$$

since they satisfy the conditions (4.2.1): (α_1, α_2) restrict to the same class in E and push forward to α and 0 in X respectively. More generally:

LEMMA 4.2. *Let $\alpha(0) = (\alpha_1, \alpha_2)$ be a choice of lifting. Then*

$$\alpha(0) = (\alpha_1 - \iota_{1*}e, \alpha_2 + \iota_{2*}e)$$

is also a lifting for any class e in E of the same dimension as α. Moreover, any two liftings are related in this manner. In particular, α_1 and α_2 are uniquely determined by each other.

Proof. The first statement follows from the facts that

$$\iota_1^*\iota_{1*}e = (e.c_1(N_{E/Y}))_E = -(e.c_1(N_{E/\tilde{E}}))_E = -\iota_2^*\iota_{2*}e$$

and $-\phi_*\iota_{1*}e + p_*\iota_{2*}e = 0$ (since $\phi \circ \iota_1 = p \circ \iota_2 = \bar{\phi} : E \to Z$).

For the second statement, let (α_1, α_2) and (a_1, a_2) be two liftings. From

$$\phi_*(\alpha_1 - a_1) = -p_*(\alpha_2 - a_2) \in H^*(Z),$$

we have that $\phi^*\phi_*(\alpha_1 - a_1)$ is a class in E. Hence $\alpha_1 - a_1 = \iota_{1*}e$ for $e \in H^*(E)$. It remains to show that if (a_1, a_2) and $(a_1, \tilde{a}_2)$ are two liftings then $a_2 = \tilde{a}_2$. Indeed by (4.2.1), $\iota_2^*(a_2 - \tilde{a}_2) = 0$. Hence by Lemma 4.3 below $z := a_2 - \tilde{a}_2 \in i_*H^*(Z)$. By (4.2.1) again $z = p_*z = p_*(a_2 - \tilde{a}_2) = 0$. $\square$

For an ordinary flop $f : X \dashrightarrow X'$, we compare the degeneration expressions of X and X'. For a given admissible triple $\eta = (\Gamma_1, \Gamma_2, I_\rho)$ on the degeneration of X, one may pick the corresponding $\eta' = (\Gamma'_1, \Gamma'_2, I'_\rho)$ on the degeneration of X' such that $\Gamma_1 = \Gamma'_1$. Since

$$\phi^*\alpha - \phi'^*\mathcal{F}\alpha \in \iota_{1*}H^*(E) \subset H^*(Y),$$

Lemma 4.2 implies that one can choose $\alpha_1 = \alpha'_1$. This can be done, for example, by modifying the choice of (4.2.2) $j_1^*\alpha(0) = \phi^*\alpha$ and $j_1'^*\mathcal{F}\alpha(0) = \phi'^*\mathcal{F}\alpha$ by adding suitable classes in E to make them equal. The above procedures identify relative invariants on the $Y_1 = Y = Y'_1$ from both sides term by term, and we are left with the comparison of the corresponding relative invariants on $\tilde{E}$ and $\tilde{E}'$. The following simple lemma is useful.

Lemma 4.3. *Let $\tilde{E} = \mathbb{P}_Z(N \oplus \mathcal{O})$ be a projective bundle with base $i : Z \hookrightarrow \tilde{E}$ and infinity divisor $\iota_2 : E = \mathbb{P}_Z(N) \hookrightarrow \tilde{E}$. Then the kernel of the restriction map $\iota_2^* : H^*(\tilde{E}) \to H^*(E)$ is $i_*H^*(Z)$.*

Proof. $i_*H^*(Z)$ obviously lies in the kernel of ι_2^*. The fact it is the entire kernel can be seen, for example, by a dimension count. □

The ordinary flop f induces an ordinary flop

$$\tilde{f} : \tilde{E} \dashrightarrow \tilde{E}'$$

on the local model. Moreover $\tilde{f}$ may be considered as a family of simple ordinary flops $\tilde{f}_t : \tilde{E}_t \dashrightarrow \tilde{E}'_t$ over the base S, where $t \in S$ and $\tilde{E}_t$ is the fiber of $\tilde{E} \to Z \to S$ etc.. Denote again by $\mathcal{F}$ the cohomology correspondence induced by the graph closure. Then

Proposition 4.4 (Cohomology reduction to local models). *Let $f : X \dashrightarrow X'$ be a $\mathbb{P}^r$ flop over base S with $\dim S = s$. Let $\alpha \in H^*(X)$ with liftings $\alpha(0) = (\alpha_1, \alpha_2)$ and $\mathcal{F}\alpha(0) = (\alpha'_1, \alpha'_2)$. Then*

$$\alpha_1 = \alpha'_1 \quad\Longleftrightarrow\quad \mathcal{F}\alpha_2 = \alpha'_2.$$

Proof. Let $\alpha \in H^{2l}(X)$ with $l \in \frac{1}{2}\mathbb{N}$. If $l > \dim Z = r + s$ then $\alpha|_Z = 0$. By (4.2.2) and Lemma 4.2, all liftings take the form $\alpha(0) = (\alpha - \iota_{1*}e, \iota_{2*}e)$ and $\mathcal{F}\alpha(0) = (\alpha - \iota'_{1*}e', \iota'_{2*}e')$ for e, e' being classes in E. In this case the proof is trivial since $\mathcal{F}$ is the identity map on $H^*(E)$. So we may assume that $l \le r + s$.

($\Rightarrow$) From the contact order condition $\iota_2^*\alpha_2 = \iota_1^*\alpha_1 = \iota_1'^*\alpha'_1 = \iota_2'^*\alpha'_2$ and the fact that $\tilde{f}$ is an isomorphism outside Z, we get

$$\iota_2'^*(\mathcal{F}\alpha_2 - \alpha'_2) = \mathcal{F}\iota_2^*\alpha_2 - \iota_2'^*\alpha'_2 = \iota_2^*\alpha_2 - \iota_2'^*\alpha'_2 = 0.$$

Thus $\mathcal{F}\alpha_2 - \alpha'_2 = i'_*z'$ for some $z' \in H^{2(l-(r+1))}(Z')$ (where $i' : Z' \hookrightarrow \tilde{E}'$) by Lemma 4.3 and the fact that $\operatorname{codim}_{\tilde{E}'} Z' = r + 1$.

For simple flops, $s = 0$ and then $l - (r+1) \le s - 1 < 0$. So $z' = 0$ and we are done. In general we restrict the equation to each fiber $\tilde{f}_t : \tilde{E}_t \to \tilde{E}'_t$. Since $\bar{\Gamma}_{\tilde{f}}|_t = \bar{\Gamma}_{\tilde{f}_t}$, by the case of simple flops we get $(\mathcal{F}\alpha_2 - \alpha'_2)|_{\tilde{E}'_t} = 0$ for all $t \in S$. That is, z' is a class supported in the fiber of $p' : Z' \to S$. But then $\operatorname{codim}_{\tilde{E}'} z' \ge s + r + 1 > l$, which implies that $z' = 0$.

($\Leftarrow$) For ease of notations we omit the imbedding maps of E into Y, $\tilde{E}$ and $\tilde{E}'$. By (4.2.2) and Lemma 4.2 we have $\alpha_1 = \phi^*\alpha - e_1$ and $\alpha'_1 = \phi'^*\mathcal{F}\alpha - e'_1$ for some classes e_1, e'_1 in E. Thus $\alpha'_1 = \alpha_1 - e$ for some class e in E. By Lemma 4.2 again $\alpha(0)$ has a lifting $(\alpha_1 - e, \alpha_2 + e) = (\alpha'_1, \alpha_2 + e)$ and by the first part of this proposition we must have $\mathcal{F}(\alpha_2 + e) = \alpha'_2$. By assumption $\mathcal{F}\alpha_2 = \alpha'_2$, hence $\mathcal{F}e = 0$ and then $e = 0$. □

Remark 4.5. *Proposition 4.4 (with cohomology groups being replaced by Chow groups) leads to an alternative proof of equivalence of Chow motives under ordinary flops. Indeed the equivalence of Chow groups for simple flops is easy to establish. The degeneration to normal cone then allows us to reduce the general case to the local case and then to the local simple case.*

4.3. Reduction to relative local models

First notice that $A_1(\tilde{E}) = \iota_{2*}A_1(E)$ since both are projective bundles over Z. We then have

$$\phi^*\beta = \beta_1 + \beta_2$$

by regarding β_2 as a class in $E \subset Y$. Indeed $\phi_*(\beta_1 + \beta_2) = \phi_*\beta_1 + p_*\beta_2 = \beta$ and

$$((\beta_1 + \beta_2).E)_Y = (\beta_1.E)_Y - (\beta_2.E)_{\tilde{E}} = |\mu| - |\mu| = 0$$

(where $N_{E/\tilde{E}} \cong N^*_{E/Y}$ is used). These characterize the class $\phi^*\beta$.

We consider only the case $g = 0$. Define the generating series

$$\langle A \mid \varepsilon, \mu\rangle^{(\tilde{E},E)} := \sum_{\beta_2 \in NE(\tilde{E})} \frac{1}{|\mathrm{Aut}\,\mu|} \langle A \mid \varepsilon, \mu\rangle^{(\tilde{E},E)}_{\beta_2}\, q^{\beta_2}.$$

and the similar one with possibly disconnected domain curves

$$\langle A \mid \varepsilon, \mu\rangle^{\bullet(\tilde{E},E)} := \sum_{\Gamma;\, \mu_\Gamma = \mu} \frac{1}{|\mathrm{Aut}\,\Gamma|} \langle A \mid \varepsilon, \mu\rangle^{\bullet(\tilde{E},E)}_{\Gamma}\, q^{\beta^\Gamma}.$$

Proposition 4.6. *To prove $\mathcal{F}\langle\alpha\rangle^X \cong \langle\mathcal{F}\alpha\rangle^{X'}$ (for all α), it is enough to show that*

$$\mathcal{F}\langle A \mid \varepsilon, \mu\rangle^{(\tilde{E},E)} \cong \langle \mathcal{F}A \mid \varepsilon, \mu\rangle^{(\tilde{E}',E)} \tag{4.3.1}$$

for all A, ε, μ.

Proof. For the n-point function $\langle\alpha\rangle^X = \sum_{\beta\in NE(X)}\langle\alpha\rangle^X_\beta\, q^\beta$, the degeneration formula gives

$$\begin{aligned}\langle\alpha\rangle^X &= \sum_{\beta\in NE(X)}\sum_{\eta\in\Omega_\beta}\sum_I C_\eta\langle\alpha_1 \mid e_I,\mu\rangle^{\bullet(Y_1,E)}_{\Gamma_1}\langle\alpha_2 \mid e^I,\mu\rangle^{\bullet(Y_2,E)}_{\Gamma_2}\, q^{\phi^*\beta}\\ &= \sum_\mu\sum_I\sum_{\eta\in\Omega_\mu} C_\eta\left(\langle\alpha_1 \mid e_I,\mu\rangle^{\bullet(Y_1,E)}_{\Gamma_1}\, q^{\beta_1}\right)\left(\langle\alpha_2 \mid e^I,\mu\rangle^{\bullet(Y_2,E)}_{\Gamma_2}\, q^{\beta_2}\right).\end{aligned}$$

To simplify the generating series, we consider also absolute invariants $\langle\alpha\rangle^{\bullet X}$ with possibly disconnected domain curves as before. Then by comparing the order of automorphisms,

$$\langle\alpha\rangle^{\bullet X} = \sum_\mu m(\mu)\sum_I\langle\alpha_1 \mid e_I,\mu\rangle^{\bullet(Y_1,E)}\langle\alpha_2 \mid e^I,\mu\rangle^{\bullet(Y_2,E)}.$$

To compare $\mathcal{F}\langle\alpha\rangle^{\bullet X}$ and $\langle\mathcal{F}\alpha\rangle^{\bullet X'}$, by Proposition 4.4 we may assume that $\alpha_1 = \alpha_1'$ and $\alpha_2' = \mathcal{F}\alpha_2$. This choice of cohomology liftings identifies the relative invariants of (Y_1, E) and those of (Y_1', E') with the same topological types. It remains to compare

$$\langle\alpha_2 \mid e^I,\mu\rangle^{\bullet(\tilde E,E)} \quad\text{and}\quad \langle\mathcal{F}\alpha_2 \mid e^I,\mu\rangle^{\bullet(\tilde E',E)}.$$

We further split the sum into connected invariants. Let Γ^π be a connected part with the contact order μ^π induced from μ. Denote $P : \mu = \sum_{\pi\in P}\mu^\pi$ a partition of μ and $P(\mu)$ the set of all such partitions. Then

$$\langle A \mid \varepsilon,\mu\rangle^{\bullet(\tilde E,E)} = \sum_{P\in P(\mu)}\prod_{\pi\in P}\sum_{\Gamma^\pi}\frac{1}{|\operatorname{Aut}\mu^\pi|}\langle A^\pi \mid \varepsilon^\pi,\mu^\pi\rangle^{(\tilde E,E)}_{\Gamma^\pi}\, q^{\beta^{\Gamma^\pi}}.$$

If one fixes the above data in the summation of (4.3.1), then the only index to be summed over is β^{Γ^π} on $\tilde E$. This reduces the problem to $\langle A^\pi \mid \varepsilon^\pi,\mu^\pi\rangle^{(\tilde E,E)}$. □

REMARK 4.7. *Here is a brief comment on the term*

$$\mathcal{F}\langle\alpha_2 \mid e^I,\mu\rangle^{(\tilde E,E)} = \sum_{\beta_2\in NE(\tilde E)}\frac{1}{|\operatorname{Aut}\mu|}\langle\alpha_2 \mid e^I,\mu\rangle^{(\tilde E,E)}_{\beta_2}\, q^{\mathcal{F}\beta_2}.$$

Since $\tilde E$ is a projective bundle, $NE(\tilde E) = i_*NE(Z)\oplus\mathbb{Z}_+\gamma$ with γ the fiber line class of $\tilde E\to Z$. The point is that, for $\beta_2\in NE(\tilde E)$ it is in general not true that $\mathcal{F}\beta_2 \equiv \beta_2$ (in E) is effective in $\tilde E'$.

Indeed, for simple ordinary flops, let $\gamma = \delta'$, $\delta = \gamma'$ be the two line classes in $E \cong \mathbb{P}^r\times\mathbb{P}^r$. It is easily checked that $\ell\sim\delta-\gamma$ in $\tilde E$. Hence $\ell = -\ell'$ and $\gamma = \gamma'+\ell'$ and

$$\beta_2 = d_1\ell + d_2\gamma = (d_2-d_1)\ell' + d_2\gamma'.$$

$\mathcal{F}\beta_2 \in NE(\tilde{E}')$ *if and only if* $d_2 \geq d_1$. *Therefore,*

$$\langle \alpha_2 \mid e^I, \mu \rangle^{(\tilde{E},E)} = \langle \mathcal{F}\alpha_2 \mid e^I, \mu \rangle^{(\tilde{E}',E)}$$

cannot possibly hold term by term. Analytic continuations are in general needed.

4.4. Relative to absolute

Recall that we are now in the local relative case, with $X = \tilde{E}$. We shall combine a method of Maulik and Pandharipande (Lemma 4 in [20]) to further reduce the relative cases to the absolute cases with at most descendent insertions along E. Following [20], we call the pair

$$(\varepsilon, \mu) = \{(\varepsilon_1, \mu_1), \cdots, (\varepsilon_\rho, \mu_\rho)\}$$

with $\varepsilon_i \in H^*(E)$, $\mu_i \in \mathbb{N}$ a *weighted partition*, a partition of contact orders weighted by cohomology classes in E.

Proposition 4.8. *For an ordinary flop* $\tilde{E} \dashrightarrow \tilde{E}'$, *to prove*

$$\mathcal{F}\langle A \mid \varepsilon, \mu \rangle \cong \langle \mathcal{F}A \mid \varepsilon, \mu \rangle$$

for any A *and* (ε, μ), *it is enough to show that*

$$\mathcal{F}\langle A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho \rangle^{\tilde{E}} \cong \langle \mathcal{F}A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho \rangle^{\tilde{E}'}$$

for any possible insertions $A \in H^*(\tilde{E})^{\oplus n}$, $k_j \in \mathbb{N} \cup \{0\}$ *and* $\varepsilon_j \in H^*(E)$. *(Here we abuse the notations and denote* $\iota_{2*}\,\varepsilon \in H^*(\tilde{E})$ *by the same symbol* ε.*)*

The rest of this subsection is devoted to the proof of this proposition which proceeds inductively on the triple $(|\mu|, n, \rho)$ in the lexicographical order with ρ in the reverse order. Given $\langle \alpha_1, \ldots, \alpha_n \mid \varepsilon, \mu \rangle$, since $\rho \leq |\mu|$, it is clear that there are only finitely many triples of lower order. The proposition holds for those cases by the induction hypothesis.

We apply degeneration to the normal cone for $Z \hookrightarrow \tilde{E}$ to get $W \to \mathbb{A}^1$. Then $W_0 = Y_1 \cup Y_2$ with $\pi : Y_1 \cong \mathbb{P}_E(\mathcal{O}_E(-1,-1) \oplus \mathcal{O}) \to E$ a $\mathbb{P}^1$ bundle and $Y_2 \cong \tilde{E}$. Denote by $E_0 = E = Y_1 \cap Y_2$ and $E_\infty \cong E$ the zero and infinity divisors of Y_1 respectively. The idea is to analyze the degeneration formula for $\langle \alpha_1, \ldots, \alpha_n, \tau_{\mu_1 - 1}\varepsilon_1, \ldots, \tau_{\mu_\rho - 1}\varepsilon_\rho \rangle^{\tilde{E}}$. We follow the procedure used in the proof of Proposition 4.6 to split the generating series of invariants with possibly disconnected domain curves, according to the contact order. For $\beta = d_1\ell + d_2\gamma \in NE(\tilde{E})$, $c_1(\tilde{E}).\beta = d_2 c_1(\tilde{E}).\gamma$, hence by the virtual dimension counting d_2 is uniquely determined for a given generating series with fixed cohomology insertions.

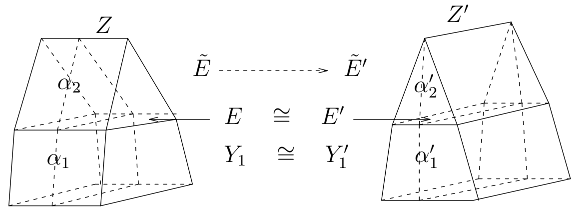

Figure 4.4.1: Degeneration to normal cone for local models.

We observe that during the splitting of β's, the "main terms" with the highest total contact order only occur when the curve classes in Y_1 are fiber classes. Indeed, let (β_1, β_2) be a splitting of β. Since

$$NE(Y_1) = \mathbb{Z}_+\delta + \mathbb{Z}_+\bar{\gamma} + \mathbb{Z}_+\gamma \quad \text{and} \quad NE(Y_2) = \mathbb{Z}_+\ell + \mathbb{Z}_+\gamma$$

($\bar{\gamma}$ is the fiber class of Y_1), we have

$$(\beta_1, \beta_2) = (a\delta + b\gamma + c\bar{\gamma}, d\ell + e\gamma)$$

subject to

$$a, b, c, d, e \geq 0, \quad a + d = d_1, \quad c = d_2$$

and the total contact order condition

$$e = (\beta_2.E)_{\tilde{E}} = (\beta_1.E)_{Y_1} = -a - b + c.$$

In particular, $e \leq d_2$ with $e = d_2$ if and only if that $a = b = 0$. In this case $\beta_1 = d_2\bar{\gamma}$ and the invariants on (Y_1, E) are fiber class integrals.

It is sufficient to consider $(\varepsilon_1, \ldots, \varepsilon_\rho) = e_I = (e_{i_1}, \ldots, e_{i_\rho})$. Since $\varepsilon_i|_Z = 0$, one may choose the cohomology lifting $\varepsilon_i(0) = (\iota_{1*}\,\varepsilon_i, 0)$. This ensures that insertions of the form $\tau_k\,\varepsilon$ must go to the Y_1 side in the degeneration formula.

Lemma 4.9. *For a general cohomology insertion $\alpha \in H^*(\tilde{E})$, the lifting can be chosen to be $\alpha(0) = (a, \alpha)$ for some a.*

Proof. $\alpha(0)$ may be chosen as $(\phi^*\alpha, p^*(\alpha|_Z))$. Since $(\alpha - p^*(\alpha|_Z)).Z = 0$, the class $e := \alpha - p^*(\alpha|_Z)$ can be taken to be supported in E. Then Lemma 4.2 implies that $\alpha(0)$ can be modified to be $(\phi^*\alpha - e, \alpha)$. □

¿From $\alpha(0) = (a, \alpha)$ and $\mathfrak{F}\alpha(0) = (a', \mathfrak{F}\alpha)$, Lemma 4.3 implies that $a = a'$. As before the relative invariants on (Y_1, E) can be regarded as constants under

$\mathcal{F}$. Then

$$\langle \alpha_1, \ldots, \alpha_n, \tau_{\mu_1 - 1} e_{i_1}, \ldots, \tau_{\mu_\rho - 1} e_{i_\rho} \rangle^{\bullet \tilde{E}} = \sum_{\mu'} m(\mu') \times$$
$$\sum_{I'} \langle \tau_{\mu_1 - 1} e_{i_1}, \ldots, \tau_{\mu_\rho - 1} e_{i_\rho} \mid e^{I'}, \mu' \rangle^{\bullet (Y_1, E)} \langle \alpha_1, \ldots, \alpha_n \mid e_{I'}, \mu' \rangle^{(\tilde{E}, E)} + R,$$

where R denotes the remaining terms which either have total contact order smaller than d_2 or have number of insertions fewer than n on the $(\tilde{E}, E)$ side or the invariants on $(\tilde{E}, E)$ are disconnected ones.

For the main terms, we claim that the total contact order $d_2 = |\mu'|$ equals $|\mu| = \sum_{i=1}^{\rho} \mu_i$. This follows from the dimension counting on $\tilde{E}$ and $(\tilde{E}, E)$. Indeed let $D = c_1(\tilde{E}).\beta + \dim \tilde{E} - 3$. For the absolute invariant on $\tilde{E}$,

$$\sum\nolimits_{j=1}^{n} \deg \alpha_j + |\mu| - \rho + \sum\nolimits_{j=1}^{\rho} (\deg e_{i_j} + 1) = D + n + \rho,$$

while on $(\tilde{E}, E)$ (notice that now $c_1(\tilde{E}).\beta_2 = d_2 c_1(\tilde{E}).\gamma = c_1(\tilde{E}).\beta$),

$$\sum\nolimits_{j=1}^{n} \deg \alpha_j + \sum\nolimits_{j=1}^{\rho'} \deg e_{i'_j} = D + n + \rho' - |\mu'|.$$

Hence (e_I, μ) appears as one of the $(e_{I'}, \mu')$'s and $|\mu| = |\mu'| = d_2$.

In particular, R is $\mathcal{F}$-invariant by induction. Moreover,

$$\deg e_I - \deg e_{I'} = \rho - \rho'.$$

We will show that the highest order term in the sum consists of the single term

$$C(\mu) \langle \alpha_1, \ldots, \alpha_n \mid e_I, \mu \rangle^{(\tilde{E}, E)}$$

where $C(\mu) \neq 0$.

For any $(e_{I'}, \mu')$ in the highest order term, consider the splitting of weighted partitions

$$(e_I, \mu) = \coprod\nolimits_{k=1}^{\rho'} (e_{I^k}, \mu^k)$$

according to the connected components of the relative moduli of (Y_1, E), which are indexed by the contact points of μ' by the genus zero assumption and the fact that the invariants on $(\tilde{E}, E)$ are connected invariants.

Since fiber class invariants on $\mathbb{P}^1$ bundles can be computed by pairing cohomology classes in E with GW invariants in the fiber $\mathbb{P}^1$ (c.f. [20], §1.2), we must have $\deg e_{I^k} + \deg e^{i'_k} \leq \dim E$ to get non-trivial invariants. That is

$$\deg e_{I^k} = \sum\nolimits_j \deg e_{i^k_j} \leq \dim E - \deg e^{i'_k} \equiv \deg e_{i'_k}$$

for each k. In particular, $\deg e_I \leq \deg e_{I'}$, hence also $\rho \leq \rho'$.

The case $\rho < \rho'$ is handled by the induction hypothesis, so we assume that $\rho = \rho'$ and then $\deg e_{I^k} = \deg e_{i'_k}$ for each $k = 1, \ldots, \rho'$. In particular $I^k \neq \emptyset$

for each k. This implies that I^k consists of a single element. By reordering we may assume that $I^k = \{i_k\}$ and $(e_{I^k}, \mu^k) = \{(e_{i_k}, \mu_k)\}$.

Since the relative invariants on Y_1 are fiber integrals, the virtual dimension for each k (connected component of the relative virtual moduli) is

$$\begin{aligned} &2\mu'_k + \dim Y_1 - 3 + 1 + (1 - \mu'_k) \\ &\quad = (\mu_k - 1) + (\deg e_{i_k} + 1) + (\dim E - \deg e_{i'_k}). \end{aligned}$$

Together with $\deg e_{i_k} = \deg e_{i'_k}$ this implies that

$$\mu'_k = \mu_k, \quad k = 1, \ldots, \rho.$$

¿From the fiber class invariants consideration and

$$\deg e_{i_k} + \deg e^{i'_k} = \dim E,$$

e_{i_k} and $e^{i'_k}$ must be Poincaré dual to get non-trivial integral over E. That is, $e_{i'_k} = e_{i_k}$ for all k and $(e_{I'}, \mu') = (e_I, \mu)$. This gives the term we expect for with $C(\mu)$ a nontrivial fiber class invariant. The proof of Proposition 4.8 is complete.

The functional equations for these special absolute invariants with descendents will be handled in §5.

4.5. Examples

We consider simple $\mathbb{P}^r$ flops for $r \le 2$ in general and for $r \ge 3$ under nefness constraint on K_X.

If $\beta = d\ell$, the invariant depends only on Z, $\alpha|_Z$ and $N_{Z/X}$. In particular

$$\langle \alpha \rangle^X_{g,n,d\ell} = \langle p^*(\alpha|_Z) \rangle^{\tilde{E}}_{g,n,d\ell}.$$

Thus we consider $\beta \ne d\ell$. Let $\alpha_i \in H^{2l_i}(X)$. By the divisor axiom, we may assume that $l_i \ge 2$ for all i.

For $\eta = (\Gamma_1, \Gamma_2, I_\rho)$ associated to (g, n, β), let d, d_{Γ_1} and d_{Γ_2} be the virtual dimension (without marked points) of stable morphisms into X and relative stable morphisms into (Y_1, E), (Y_2, E) respectively. We have $l_1 + \cdots + l_n = d + n$. Moreover, since $\dim E = 2r$, the degeneration formula implies that $d = d_{\Gamma_1} + d_{\Gamma_2} - 2r\rho$.

We assume that the summand given by η is not zero. Since $\beta \ne d\ell$ and $A_1(Y_2)$ is spanned by ℓ and a fiber line γ, we see that $\beta_1 \ne 0$ and $\Gamma_1 \ne \emptyset$.

If $\rho = 0$ then $\Gamma_2 = \emptyset$ by connectedness, and this gives the blow-up term

$$\langle \tilde{\alpha} \rangle^Y_{g,n,\phi^*\beta}.$$

So we assume that $\rho \ne 0$. By reordering, we may assume that in the degeneration expression α_i appears in the Y_1 part for $1 \le i \le m$ and α_i appears in the Y_2 part for $m+1 \le i \le n$. By transversality, the corresponding relative invariant is non-trivial only if $2 \le l_i \le r$ for $m+1 \le i \le n$. If $r = 1$ this simply means that all α_i's appear in Y_1. In the following we abuse the notation by writing $|\mu|$ as μ.

THEOREM 4.10 (Li-Ruan [15]). *For simple $\mathbb{P}^1$ flops of threefolds with $\beta \ne d\ell$,*

$$\langle \alpha \rangle^X_{g,n,\beta} = \langle \tilde{\alpha} \rangle^Y_{g,n,\phi^*\beta} = \langle \mathcal{F}\alpha \rangle^{X'}_{g,n,\mathcal{F}\beta}.$$

That is, there are no degenerate terms and hence no analytic continuations are needed for non-exceptional curve classes.

Proof. If $r = 1$, then $(K_X.p_*\beta_2) = 0$, $d = -(K_X.\beta)$ and

$$\begin{aligned}(K_Y.\beta_1) &= (\phi^* K_X.\beta_1) + (E.\beta_1) = (K_X.\phi_*\beta_1) + \mu \\ &= (K_X.(\beta - p_*\beta_2)) + \mu = (K_X.\beta) + \mu.\end{aligned}$$

So

$$d_{\Gamma_1} = -(K_Y.\beta_1) + \rho - \mu = d + (\rho - 2\mu).$$

If $\rho \ne 0$ then $d_{\Gamma_1} < d$. Since $l_i \ge 2$, we may assume that α_i's are disjoint from Z, hence they must all contribute to the Y_1 part. This forces that $\rho = 0$ and the result follows. $\square$

For simple $\mathbb{P}^2$ flops, non-trivial degenerate terms do occur even for $n \le 3$ and $g = 0$. Let $v_i := |\Gamma_i|$ be the number of connected components.

LEMMA 4.11. *For $\tilde{E} = \mathbb{P}_Z(N \oplus \mathcal{O})$ of a pair $Z \subset X$,*

$$c_1(\tilde{E}) = (\operatorname{rk} N + 1)E + p^* c_1(X)|_Z.$$

Proof. Indeed, from $0 \to \mathcal{O} \to \mathcal{O}(1) \otimes p^*(N \oplus \mathcal{O}) \to T_{\tilde{E}/Z} \to 0$ we get $c_1(T_{\tilde{E}/Z}) = (\operatorname{rk} N + 1)E + p^* c_1(N)$, so the formula follows from $c_1(\tilde{E}) = c_1(T_{\tilde{E}/Z}) + p^* c_1(Z)$. $\square$

PROPOSITION 4.12. *For simple $\mathbb{P}^2$ flops, let $n \le 3$ and $\alpha_i \in H^{2l_i}(X)$ with $l_i \ge 2$ for $i = 1, \ldots, n$. Consider $\beta \ne d\ell$ and an admissible triple η with $\rho \ne 0$. Then $v_1 = \rho = \mu$, $v_2 = 1$ and $l_i = 2$ for all i.*

Proof. Since $c_1(Y_2) = 4E$ (by Lemma 4.11), we find that

$$\begin{aligned}d_{\Gamma_2} &= 4(E.\beta_2) + 2v_2 + \rho - \mu \\ &= 3\mu + \rho + 2v_2.\end{aligned}$$

So $d_{\Gamma_2} - 4\rho = 3(\mu - \rho) + 2v_2 \ge 2$.

For one-point invariants, $l_1 = d+1 = d_{\Gamma_1} + 3(\mu - \rho) + 2v_2 + 1 \geq d_{\Gamma_1} + 3$. It forces that α_1 contributes in Y_2, hence $l_1 = 2$ and $d = 1$. But $d_{\Gamma_1} \geq 0$ implies that $d \geq 2$, hence a contradiction.

For two-point invariants, from $l_1 + l_2 = d + 2 = d_{\Gamma_1} + 3(\mu - \rho) + 2v_2 + 2 \geq d_{\Gamma_1} + 4$ and the fact that α_i contributes to the Y_2 part in the degeneration formula only if $l_i = 2$, similar argument shows that the only non-trivial case is that $l_1 = l_2 = 2$ and both α_1 and α_2 contribute in Y_2. Moreover the equality holds hence that $\mu = \rho$, $v_2 = 1$ and $d_{\Gamma_1} = 0$.

We now consider three-point invariants. From

$$l_1 + l_2 + l_3 = d + 3 = d_{\Gamma_1} + (d_{\Gamma_2} - 4\rho) + 3 \geq d_{\Gamma_1} + 5,$$

if only α_3 contributes to Y_2 then $l_1 + l_2 \geq d_{\Gamma_1} + 3 > d_{\Gamma_1} + 2$ leads to trivial invariant. If α_2 and α_3 contribute to Y_2, then $l_1 \geq d_{\Gamma_1} + 1$. This leads to non-trivial invariant only if equality holds. That is, $\mu = \rho$ and $v_2 = 1$.

The remaining case is that $l_i = 2$, α_i contributes in Y_2 for all $i = 1, 2, 3$. We have $\mu = \rho$, $v_2 = 1$, $d = 3$, $d_{\Gamma_1} = 1$, $d_{\Gamma_2} = 4\rho + 2$. □

To summarize, notice that the weighted partitions associated to the relative invariants on the $Y_2 = \tilde{E}$ part are of the form $(\mu_1, \ldots, \mu_n) = (1, \ldots, 1)$ and $\deg \alpha_i = 2$ for all i, thus they are of the lowest order with fixed $|\mu|$. They can be reduced to absolute invariants readily.

For $\beta_2 = d_1\ell + d_2\gamma$, we see that $d_2 = \mu = \rho$ and so

$$d_{\Gamma_2} = 4d_2 + 2$$

is independent of d_1. Also d_2 is uniquely determined by the cohomology insertions. The presence of degenerate terms with degree β_2 for all large d_1 indicates the necessity of analytic continuations. (c.f. Example 5.7.)

The same conclusion holds for $r \geq 3$ if we impose the nefness of K_X. We state the result in a slightly more general form:

Proposition 4.13. *Let $\phi : Y \to X$ be the blow-up of X along a smooth center Z of dimension r and codimension $r' + 1$ with K_X nef and $r \leq r' + 1$. Then $C_\eta \neq 0$ only if $g_1 = 0$, $v_1 = \mu = \rho \neq 0$ and $\mu_1 \equiv 1$, $v_2 = 1$.*

The proof is entirely similar and we omit it.

5. Analytic Continuations on Local Models

The basic strategy to calculate GW invariants on local models is similar to §3. Here we start with one point invariants on toric varieties. The compatibility of functional equations under the reconstruction procedure is proved with help from operators δ_H's which generalize $q^\ell d/dq^\ell$, the one used in Corollary 3.2.

5.1. One-point functions on local models

In this section X is the local model

$$X = \mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(-1)^{\oplus(r+1)} \oplus \mathcal{O}).$$

The cohomology (Chow ring) is given by

$$H^*(X) = A^*(X) = \mathbb{Z}[h, \xi]/(h^{r+1}, (\xi - h)^{r+1}\xi).$$

Since $c_1(X) = (r+2)\xi$ is semi-positive, X is a semi-Fano toric variety.

The toric fan $\triangle(X)$ of X is given by one dimensional edges

$$w_0, \ldots, w_{r+1}, v_0, \ldots, v_r \in \mathbb{Z}^{r+(r+1)}$$

such that

$$w_0 + w_1 + \cdots + w_{r+1} = 0, \qquad v_0 + \cdots + v_r = w_0 + \cdots + w_r = -w_{r+1}.$$

Let $\{e_i\}_{i=0,\ldots,r-1}$ and $\{e'_i\}_{i=0,\ldots,r}$ be the basis of $\mathbb{Z}^r \times \mathbb{Z}^{r+1}$. Then we may pick

$$w_i = e'_i, \quad 0 \le i \le r; \qquad w_{r+1} = -e'_0 - \cdots - e'_r;$$

$$v_i = e_i + e'_i, \quad 0 \le i \le r-1; \qquad v_r = -e_0 - \cdots - e_{r-1} + e'_r.$$

This implies the following linear equivalence of toric divisors

$$D_{v_0} = D_{v_1} = \cdots = D_{v_r} =: h; \qquad \xi := D_{w_{r+1}} = D_{w_i} + D_{v_i}, \quad i = 0, \ldots, r.$$

Thus $D_{w_i} = \xi - h$ for all $i = 0, \ldots, r$.

Remark 5.1. *In terms of the homogeneous coordinate rings, X is defined by an embedding of $(\mathbb{C}^*)^2 \hookrightarrow (\mathbb{C}^*)^{2r+1}$, which is defined by the $2\times(2r+1)$ matrix $M : \mathrm{Lie}(\mathbb{C}^*)^{2r+1} \to \mathrm{Lie}(\mathbb{C}^*)^2$*

$$M = \begin{pmatrix} 1 & \ldots & 1 & -1 & \ldots & -1 & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 1 & 1 \end{pmatrix},$$

where on the first row, there are r 1's, r (-1)'s. The Kähler cone is spanned by h and ξ on $H^2(X) \cong \mathbb{C}^2$.

We start with one-point descendent invariants. The toric data allows us to apply the known results of [7] [17] directly. Let

$$P_\beta := \frac{\prod_{\rho\in\triangle_1(X)} \prod_{m=-\infty}^{0} (D_\rho + mz)}{\prod_{\rho\in\triangle_1(X)} \prod_{m=-\infty}^{(\beta.D_\rho)} (D_\rho + mz)}.$$

LEMMA 5.2. *For an effective curve class $\beta = d_1\ell + d_2\gamma$,*

$$J_X(\beta, z^{-1}) = \frac{\prod_{m=-\infty}^{0} (\xi - h + mz)^{r+1}}{\prod_{m=1}^{d_1} (h + mz)^{r+1} \prod_{m=-\infty}^{d_2-d_1} (\xi - h + mz)^{r+1} \prod_{m=1}^{d_2} (\xi + mz)}.$$

Proof. Since $(\beta.h) = d_1$ and $(\beta.\xi) = d_2$, the right hand side is precisely P_β. $J_X(\beta, z^{-1})$ is equal to P_β without change of variables ("mirror transformation") due to the uniqueness theorem and the fact that $P_\beta = O(1/z^2)$ in $1/z$ power series expansion. Indeed if $d_1 \le d_2$,

$$P_\beta = \frac{1}{(d_1!)^{r+1}((d_2 - d_1)!)^{r+1} d_2!} \frac{1}{z^{d_2(r+2)}} + \cdots,$$

while if $d_1 > d_2$ (the key observation),

$$P_\beta = (\xi - h)^{r+1} \Big(\frac{((d_1 - d_2 - 1)!)^{r+1}}{(d_1!)^{r+1} d_2!} (-1)^{d_1 - d_2 - 1} \frac{1}{z^{d_2(r+2)+r+1}} + \cdots \Big).$$

For more details see [7] [17]. □

It also follows that a presentation of the small quantum cohomology ring is given by Batyrev's quantum ring (cf. [5], the proof of Proposition 11.2.17). Namely for $q_1 = q^\ell$ and $q_2 = q^\gamma$,

$$QH^*(X) = \mathbb{C}[h, \xi][\![q_1, q_2]\!]/(h^{r+1} - q_1(\xi - h)^{r+1}, (\xi - h)^{r+1}\xi - q_2).$$

Though the presentation does not provide enough information for our purpose, it does give a first test of the invariance property.

PROPOSITION 5.3. *The map $\mathcal{F}_X : QH^*(X)[q_1^{-1}] \to QH^*(X')[q_1'^{-1}]$ defined by $\mathcal{F}_X h = \xi' - h'$, $\mathcal{F}_X \xi = \xi'$, $\mathcal{F}_X q_1 = q_1'^{-1}$ and $\mathcal{F}_X q_2 = q_1' q_2'$ extends to a ring isomorphism.*

Proof. Since $\mathcal{F}_{X'} \circ \mathcal{F}_X = \mathrm{Id}_X$, it is enough to check that the generators of the ideal are mapped into the corresponding ideal in the X' side:

$$\begin{aligned} \mathcal{F}_X(h^{r+1} - q_1(\xi - h)^{r+1}) &= (\xi' - h')^{r+1} - q_1'^{-1} h'^{r+1} \\ &= -q_1'^{-1}(h'^{r+1} - q_1'(\xi' - h')^{r+1}); \end{aligned}$$

$$\begin{aligned} \mathcal{F}_X((\xi - h)^{r+1}\xi - q_2) &= h'^{r+1}\xi' - q_1' q_2' \\ &= (h'^{r+1} - q_1'(\xi' - h')^{r+1})\xi' + q_1'((\xi' - h')^{r+1}\xi' - q_2'). \end{aligned}$$

□

Note that the virtual dimension of an n-point invariants in degree $\beta = d_1\ell + d_2\gamma$ is given by $D_{n,\beta} = (r+2)d_2 + 2r + n - 2$, so for a fixed set of cohomology insertions there could be at most one d_2 supporting non-trivial invariants and for the corresponding n-point function the summation over d_2 is unnecessary.

LEMMA 5.4 (Quasi-linearity). *Let $J_X := J_X(q, z^{-1})$. For any $\alpha \in H^*(X)$, the one point function $\langle \tau_k \xi \alpha \rangle^X$ satisfies the functional equation (without analytic continuation):*

$$\mathcal{F} \langle \tau_k \xi.\alpha \rangle^X = \langle \tau_k \mathcal{F}(\xi.\alpha) \rangle^{X'} = \left\langle \tau_k \xi'.\mathcal{F}\alpha \right\rangle^{X'}.$$

Equivalently, $\mathcal{F}$ is linear in $J\xi$:

$$\mathcal{F}(J_X \xi.\alpha) = J_{X'} \mathcal{F}(\xi.\alpha) = J_{X'} \xi'.\mathcal{F}\alpha.$$

Proof. The key observation on P_β is that if $d_2 - d_1 < 0$ then the middle factor in the denominator of P_β goes to the numerator instead which has a factor $(\xi - h)^{r+1}$. Thus it vanishes after multiplication by ξ. Notice that the condition $d_2 \geq d_1$ simply corresponds to the effectivity of $\mathcal{F}\beta = -d_1\ell' + d_2(\gamma' + \ell') = (d_2 - d_1)\ell' + d_2\gamma'$.

Since $J_X = \sum_{\beta \in NE(X)} q^\beta P_\beta$, by the above observation $J_X \xi.\alpha$ can be written as

$$J_X \xi.\alpha = \sum_{d_2} \frac{1}{\prod_{m=1}^{d_2} (\xi + mz)} \sum_{d_1=0}^{d_2} \frac{q^{d_2\gamma} q^{d_1\ell}.\xi.\alpha}{\prod_{m=1}^{d_1} (h + mz)^{r+1} \prod_{m=1}^{d_2-d_1} (\xi - h + mz)^{r+1}}.$$

Notice that since the flop is an isomorphism outside $Z = \mathbb{P}^r \subset X$, the cohomology correspondence $\mathcal{F}$ is the "identity" one on classes $\xi.\alpha$. Namely $\mathcal{F}h^i = (\xi' - h')^i$ for $i \leq r$ and $\mathcal{F}(\xi.\alpha) = \mathcal{F}\xi.\mathcal{F}\alpha = \xi'.\mathcal{F}\alpha$ for any $\alpha \in H^*(X)$, Thus

$$\mathcal{F}(J_X \xi.\alpha) = \sum_{d_2} \frac{1}{\prod_{m=1}^{d_2} (\xi' + mz)} \sum_{d_1=0}^{d_2} \frac{q^{d_2(\gamma'+\ell')} q^{-d_1\ell'}.\xi'.\mathcal{F}\alpha}{\prod_{m=1}^{d_1} (\xi' - h' + mz)^{r+1} \prod_{m=1}^{d_2-d_1} (h' + mz)^{r+1}}.$$

By rewriting the inner summation to be on $d_1' = d_2 - d_1 \in \{0, \ldots, d_2\}$ we arrive at the corresponding expression of $J_{X'}\xi'.\mathcal{F}\alpha$.

Since for given insertion(s) there could be at most one d_2 supporting non-trivial invariants, we find that $\langle \tau_k \xi \alpha \rangle$ is a finite sum and $\mathcal{F} \langle \tau_k \xi.\alpha \rangle = \langle \tau_k \xi'.\mathcal{F}\alpha \rangle$ holds without the need of analytic continuation. $\square$

5.2. The functional equations in general

Write $\beta = d_1\ell + d_2\gamma$. If $d_2 = 0$, the whole setting on Gromov-Witten invariants goes back to quantum corrections attached to the extremal ray $\mathbb{Z}\ell$. In §3 we had seen that while n-point functions with $n \ge 3$ satisfy the functional equation under $\mathcal{F}$ up to analytic continuation, it is not the case for $n = 2$ or descendent invariants with $n - 3 - k < 0$.

The results in §2 and the quasi-linearity lemma are the induction basis of our discussion on functional equations up to analytic continuation.

For a power series $f = \sum_\beta a_\beta\, q^\beta$ and a divisor H, we define the operator

$$\delta_H f := \sum_\beta (H.\beta) a_\beta\, q^\beta = \Big((H.\ell) q^\ell \frac{\partial}{\partial q^\ell} + (H.\gamma) q^\gamma \frac{\partial}{\partial q^\gamma}\Big) f.$$

The following lemma formalizes the argument in the proof of Corollary 3.2:

Lemma 5.5. *The differential operator δ_H is $\mathcal{F}$ equivariant. That is,*

$$\mathcal{F} \circ \delta_H = \delta_{\mathcal{F}H} \circ \mathcal{F}.$$

In particular, if $\mathcal{F}\langle\alpha\rangle \cong \langle\mathcal{F}\alpha\rangle$ then $\mathcal{F}\delta_H\langle\alpha\rangle \cong \delta_{\mathcal{F}H}\langle\mathcal{F}\alpha\rangle$ too.

Proof. This follows from the fact that $\mathcal{F}$ preserves the Poincaré pairing. In explicit terms, denote by $(x, y) = (q^\ell, q^\gamma)$ and $(x', y') = (q^{\ell'}, q^{\gamma'})$ respectively. The transformation law $x' = x^{-1}$, $y' = xy$ leads to

$$x\frac{\partial}{\partial x} = -x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'}; \qquad y\frac{\partial}{\partial y} = y'\frac{\partial}{\partial y'}.$$

Hence

$$\begin{aligned}\mathcal{F} \circ \delta_H &= (\mathcal{F}H.\mathcal{F}\ell)\Big(-x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'}\Big) + (\mathcal{F}H.\mathcal{F}\gamma) y'\frac{\partial}{\partial y'} \\ &= (\mathcal{F}H.\ell') x'\frac{\partial}{\partial x'} + (\mathcal{F}H.\mathcal{F}(\gamma + \ell)) y'\frac{\partial}{\partial y'} = \delta_{\mathcal{F}H} \circ \mathcal{F}.\end{aligned}$$

If $\mathcal{F}\langle\alpha\rangle \cong \langle\mathcal{F}\alpha\rangle$ then $\mathcal{F}\delta_H\langle\alpha\rangle = \delta_{\mathcal{F}H}\mathcal{F}\langle\alpha\rangle \cong \delta_{\mathcal{F}H}\langle\mathcal{F}\alpha\rangle$. □

Theorem 5.6. *Let $\langle\alpha\rangle = \langle\alpha_1, \ldots, \alpha_n\rangle$ with $\alpha_i \in H^*(X) \cup \tau_\bullet H^*(E)$. If $d_2 \ne 0$ then*

$$\mathcal{F}\langle\alpha\rangle \cong \langle\mathcal{F}\alpha\rangle.$$

Proof. We will prove the theorem by induction on d_2 and then n. This is based on the following observations: (1) By the virtual dimension count, each set of insertions can support at most one d_2. (2) Under divisor relations the degree β is either preserved or split into effective classes $\beta = \beta_1 + \beta_2$, so d_2 is split accordingly as $d_2 = d_2^L + d_2^R$. (3) When summing over $\beta \in NE(X)$, the splitting terms can usually be written as the product of two generating series with no more marked points in a manner which will be clear in each context during the proof.

For $d_2 = 0$, since $\xi|_Z = 0$ we get trivial invariant if one of the insertions involves ξ. Hence by §3 the statement in the theorem holds for $d_2 = 0$ except for the unique case $\langle h^r, h^r\rangle$. In this case, by the divisor axiom

$$\delta_h \langle h^r, h^r\rangle = \langle h, h^r, h^r\rangle,$$

which satisfies the functional equation up to analytic continuation, as we had shown before through explicit formulae incorporated with classical defect. Thus we may base our induction on $d_2 = 0$ with special care on this case.

Let $d_2 \geq 1$. The case $n = 1$ is contained in Lemma 5.4, so let $n \geq 2$. We may and will make one more assumption that ξ appears in some α_i. If not, then there will be no descendent insertions and we may write

$$\langle \alpha_1, \ldots, \alpha_n\rangle = \langle \alpha_1, \ldots, \alpha_n, \xi\rangle / d_2$$

by the divisor axiom. In the following reduction steps, each term will either have smaller d_2 or with this condition being preserved.

By reordering we may assume that $\alpha_n = \tau_s \xi a$, $s \geq 0$. Write $\alpha_1 = \tau_k h^l \xi^j$. The induction procedure is to move divisors in α_1 into α_n in the order of ψ, h and ξ. That is we use induction on the following five numbers in the alphabetical order:

$$(d_2, n, k, l, j).$$

For ψ we use equation $\psi_1 = -\psi_n + [D_{1|n}]^{virt}$. If $k \geq 1$ then $j \neq 0$ and we get

$$\begin{aligned} \langle \tau_k h^l \xi^j, \ldots, \tau_s \xi a\rangle = {} & -\langle \tau_{k-1} h^l \xi^j, \ldots, \tau_{s+1} \xi a\rangle \\ & + \sum_i \langle \tau_{k-1} h^l \xi^j, \ldots, T_i\rangle \langle T^i, \ldots, \tau_s \xi a\rangle. \end{aligned}$$

For each i, if one of d_2^L and d_2^R is zero then since both terms contain ξ classes the splitting term must vanish. So we may assume that $d_2^L < d_2$ and $d_2^R < d_2$ and these terms are done by the induction hypothesis. By performing this procedure to $\alpha_1, \ldots, \alpha_{n-1}$ we may assume that the only descendent insertion is α_n.

For h, if $l \geq 2$ or $l = 1$ but $j \neq 0$ we use (3.4.1) to get

$$\begin{aligned} \langle h^l \xi^j, \ldots, \tau_s \xi a\rangle = {} & \langle h^{l-1} \xi^j, \ldots, \tau_s \xi a h\rangle + \delta_h \langle h^{l-1} \xi^j, \ldots, \tau_{s+1} \xi a\rangle \\ & - \sum_i \delta_h \langle h^{l-1} \xi^j, \ldots, T_i\rangle \langle T^i, \ldots, \tau_s \xi a\rangle. \end{aligned}$$

The only case for the splitting term to have one factor to have the same d_2 and n is of the form

$$\delta_h \langle h^{l-1}\xi^j, T_i\rangle\langle T^i, \alpha_2, \ldots, \alpha_{n-1}, \tau_s\xi a\rangle,$$

where the two-point invariant has $d_2^L = 0$. But then $l-1 < r$ forces it to vanish. We remark here that the case $d_2^L = 0$ may still support nontrivial invariants with three or more points if $j = 0$.

By induction (and Lemma 5.5) we are left with the case $\alpha_1 = h$. The divisor axiom implies that

$$\langle h, \ldots, \tau_s\xi a\rangle = \delta_h\langle \ldots, \tau_s\xi a\rangle + \langle \ldots, \tau_{s-1}\xi a h\rangle.$$

Since both terms have one less marked points, they are done by induction.

For ξ, the argument is entirely similar. For $j \geq 2$, the divisor relation says that

$$\begin{aligned}\langle \xi^j, \ldots, \tau_s\xi a\rangle = \langle \xi^{j-1}, \ldots, \tau_s\xi^2 a\rangle + \delta_\xi\langle \xi^{j-1}, \ldots, \tau_{s+1}\xi a\rangle \\ - \sum_i \delta_\xi\langle \xi^{j-1}, \ldots, T_i\rangle\langle T^i, \ldots, \tau_s\xi a\rangle.\end{aligned}$$

We then have $d_2^L < d_2$ and $d_2^R < d_2$ as before. If $j = 1$ we get

$$\langle \xi, \ldots, \tau_s\xi a\rangle = \delta_\xi\langle \ldots, \tau_s\xi a\rangle + \langle \ldots, \tau_{s-1}\xi^2 a\rangle$$

and both terms have fewer marked points. The proof is complete. □

Practically the above inductive procedure leads to explicit determination of GW invariants, though the computations are somewhat tedious. For the interested readers, we list the results for the two typical series of examples of the local model of simple $\mathbb{P}^2$ flop.

EXAMPLE 5.7. *Simple $\mathbb{P}^2$ flop with $d_2 = 1$, $n = 3$. The virtual dimension is 9. Then on X ($q_1 = q^\ell$, $q_2 = q^\gamma$),*

$$\langle h^2, h^2, h^2\xi^3\rangle = \frac{q_1^2}{1+q_1}q_2, \quad \langle \xi^2, \xi^2, h^2\xi^3\rangle = (1+q_1)q_2,$$

$$\langle h\xi, h\xi, h^2\xi^3\rangle = \langle h\xi, \xi^2, h^2\xi^3\rangle = \langle h\xi, h^2, h^2\xi^3\rangle = \langle \xi^2, h^2, h^2\xi^3\rangle = q_1 q_2.$$

Similar formulae hold on X'. We compute ($q'_1 = q^{\ell'}$, $q'_2 = q^{\gamma'}$)

$$\mathcal{F}\langle h^2, h^2, h^2\xi^3\rangle = \frac{q_1'^{-2}}{1+q_1'^{-1}}q'_1 q'_2 = \frac{1}{1+q'_1}q'_2;$$

$$\begin{aligned}\langle \mathcal{F}h^2, \mathcal{F}h^2, \mathcal{F}h^2\xi^3\rangle &= \langle (\xi' - h')^2, (\xi' - h')^2, \mathcal{F}[pt]\rangle \\ &= \langle \xi'^2, \xi'^2, [pt]\rangle + 4\langle \xi' h', \xi' h', [pt]\rangle + \langle h'^2, h'^2, [pt]\rangle \\ &\quad - 4\langle \xi' h', h'^2, [pt]\rangle - 4\langle \xi' h', \xi'^2, [pt]\rangle + 2\langle \xi'^2, h'^2, [pt]\rangle \\ &= \Big((1 + q_1') + 4q_1' + \frac{q_1'^2}{1 + q_1'} - 4q_1' - 4q_1' + 2q_1'\Big) q_2' \\ &= \Big(1 - q_1' + \frac{q_1'^2}{1 + q_1'}\Big) q_2' = \frac{1}{1 + q_1'} q_2'.\end{aligned}$$

Thus $\mathcal{F}\langle h^2, h^2, h^2\xi^3\rangle \cong \langle \mathcal{F}h^2, \mathcal{F}h^2, \mathcal{F}h^2\xi^3\rangle$. *We leave the simpler verifications on the other five cases to the readers.*

EXAMPLE 5.8. *Descendent invariants for simple* $\mathbb{P}^2$ *flop with* $d_2 = 1$ *and* $n = 3$.

$$\langle h^2, h^2, \tau_4\xi\rangle = 3q_1q_2 - 6\frac{q_1q_2}{1 + q_1}, \quad \langle \xi^2, \xi^2, \tau_4\xi\rangle = 9q_2 + 9q_1q_2,$$

$$\langle h\xi, h\xi, \tau_4\xi\rangle = \langle h^2, \xi^2, \tau_4\xi\rangle = 3q_2,$$

$$\langle h\xi, h^2, \tau_4\xi\rangle = 0, \quad \langle h\xi, \xi^2, \tau_4\xi\rangle = 6q_2 + 3q_1q_2.$$

We omit the elementary verifications on functional equations.

6. Mukai flops

6.1. Twisted Mukai flops

Consider a flopping contraction of *twisted Mukai type*, namely $\psi : (X, Z) \to (\bar{X}, S)$ with $Z = \mathbb{P}_S(F) \to S$, $\operatorname{rank} F = r + 1$ and $N_{Z/X} = T^*_{Z/S} \otimes \bar{\psi}^* L$ for some twisting line bundle $L \in \operatorname{Pic}(S)$. To construct the flop, as in the case of ordinary flops, it is natural to consider the blow-up $\phi : Y = \operatorname{Bl}_Z X \to X$ and try to contract the exceptional set E in another fiber direction. We assume that $\operatorname{codim}_X Z = r \geq 2$ to exclude trivial cases.

PROPOSITION 6.1. *Twisted Mukai flops exist.*

Proof. It is well known from the case of simple Mukai flops that the fiber E_s of $E \to S$ is the degree $(1, 1)$ hypersurface $H_{1,1} \subset \mathbb{P}^r \times \mathbb{P}^r$ defined by

$$\sum_{i=0}^{r} x_i y_i = 0$$

with $N_{E/Y}|_{E_s} = \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^r}(-1, -1)|_{H_{1,1}}$ (cf. [9] or the constructions given below). With this, the same proof as in Proposition 1.3 works here. Indeed, under the same notations, we take C_Y to be a line in the $\mathbb{P}^{r-1}$ fiber of any $E_s \cong$

$H_{1,1} \to \mathbb{P}^r$ in the second projection. It is clear that $(K_X.C) = 0$ and then $(K_Y.C_Y) = -(r-1) < 0$. To see that C_Y is extremal, for $c = (H.C)$,

$$\mathcal{O}_Y(\phi^*H + cE)|_{E_s} \cong \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^r}(0, -c)|_{H_{1,1}}$$

and $k\phi^*L - (\phi^*H + cE)$ is a supporting divisor for $[C_Y]$ when k is large. $\square$

The proof suggests studying twisted Mukai flops via ordinary flops. Indeed, we will construct the local model of it as a slice of the ordinary flop with $F' = F^* \otimes L$. This is fundamental throughout our later discussions.

We start with an arbitrary pair (F, F') of vector bundles of rank $r+1$ and denote the corresponding maps in the ordinary $\mathbb{P}^r$ flop by $\Phi : \mathcal{Y} \to \mathcal{X}$, $\Phi' : \mathcal{Y} \to \mathcal{X}'$, $\Psi : \mathcal{X} \to \bar{\mathcal{X}}$ and $\Psi' : \mathcal{X}' \to \bar{\mathcal{X}}$. Also let $g = \Psi \circ \Phi = \Psi' \circ \Phi'$. The restriction maps on the exceptional sets are denoted by $\bar{\phi}$, $\bar{\phi}'$, $\bar{\psi}$, $\bar{\psi}'$ and $\bar{g}$ respectively.

$$\begin{tikzcd}
& \mathcal{E} = \mathbb{P}_S(F) \times_S \mathbb{P}_S(F') \subset \mathcal{Y} \arrow[dl, "\Phi"'] \arrow[dr, "\Phi'"] \arrow[dd, "g"] & \\
Z = \mathbb{P}_S(F) \subset \mathcal{X} \arrow[dr, "\Psi"'] & & Z' = \mathbb{P}_S(F') \subset \mathcal{X}' \arrow[dl, "\Psi'"] \\
& S \subset \bar{\mathcal{X}} &
\end{tikzcd}$$

First suppose that there exists a non-degenerate bilinear map

$$F \times_S F' \to \eta_S$$

with $\eta_S \in \mathrm{Pic}(S)$. (This happens precisely when $F' \cong F^* \otimes \eta_S$ for some line bundle η_S.) The map $\mathcal{O}_{\mathbb{P}(F)}(-1) \to \bar{\psi}^*F$ pulls back to $\bar{\phi}^*\mathcal{O}_{\mathbb{P}(F)}(-1) \to \bar{g}^*F$, hence leads to a natural map

$$\mathcal{O}_{\mathcal{E}}(-1,-1) := \bar{\phi}^*\mathcal{O}_Z(-1) \otimes_{\mathcal{E}} \bar{\phi}'^*\mathcal{O}_{Z'}(-1) \to \bar{g}^*(F \otimes_S F') \to \bar{g}^*\eta_S.$$

Notice that the normal bundle $N_{\mathcal{E}/\mathcal{Y}}$ equals $\mathcal{O}_{\mathcal{E}}(-1,-1)$. That is, $\mathcal{Y}$ is the total space of $\mathcal{O}_{\mathcal{E}}(-1,-1)$. Let $p : N_{\mathcal{E}/\mathcal{Y}} \to \mathcal{E}$ be the projection map.

We describe two equivalent ways to construct the space Y. The above linear map between line bundles induces a surjective map of invertible sheaves which fits into an exact sequence of the form

$$0 \to N_{\mathcal{E}/\mathcal{Y}}(-E) \to N_{\mathcal{E}/\mathcal{Y}} \to \bar{g}^*\eta_S \to 0$$

for an effective divisor $E \subset \mathcal{E}$. We then take $Y = p^{-1}(E) \subset \mathcal{Y}$ to be the collection of lines with origins in E. Alternatively Y is simply the irreducible component of the inverse image of the zero section of $\bar{g}^*\eta_S$ in $\mathcal{Y}$ other than the zero section $\mathcal{E}$.

Let $X = \Phi(Y) \supset Z$, $X' = \Phi'(Y) \supset Z'$, $\bar{X} = g(Y) \supset S$ with restriction maps ϕ, ϕ', ψ, ψ'. By tensoring the Euler sequence

$$0 \to \mathcal{O}_Z(-1) \to \bar{\psi}^* F \to \mathcal{Q} \to 0$$

with $\mathcal{S}^* = \mathcal{O}_Z(1)$ and noticing that $\mathcal{S}^* \otimes \mathcal{Q} \cong T_{Z/S}$, we get by duality

$$0 \to T^*_{Z/S} \to \mathcal{O}_Z(-1) \otimes \bar{\psi}^* F^* \to \mathcal{O}_Z \to 0.$$

The inclusion maps $Z \hookrightarrow X \hookrightarrow \mathcal{X}$ leads to

$$0 \to N_{Z/X} \to N_{Z/\mathcal{X}} \to N_{X/\mathcal{X}}|_Z \to 0.$$

Here $N_{X/\mathcal{X}}|_Z = \mathcal{O}(X)|_Z = \bar{\psi}^* \mathcal{O}(\bar{X})|_S$. Denote $\mathcal{O}(\bar{X})|_S$ by L. Recall that $N_{Z/\mathcal{X}} \cong \mathcal{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\psi}^* F'$. By tensoring with $\bar{\psi}^* L^*$, we get

$$0 \to N_{Z/X} \otimes \bar{\psi}^* L^* \to \mathcal{O}_{\mathbb{P}_S(F)}(-1) \otimes \bar{\psi}^* (F' \otimes L^*) \to \mathcal{O}_Z \to 0.$$

So $F' = F^* \otimes L$ if and only if $N_{Z/X} \cong T^*_{Z/S} \otimes \bar{\psi}^* L$.

This is the case for twisted Mukai flops. We have then $\eta_S \cong L$.

6.2. Mukai flops as limits of isomorphisms

For Mukai flops, namely $L \cong \mathcal{O}_S$, we have $F' = F^*$ with duality pairing $F \times_S F^* \to \mathcal{O}_S$.

Consider $\pi : \mathcal{Y} \to \mathbb{C}$ via

$$\mathcal{Y} \to \bar{g}^* \mathcal{O}_S = \mathcal{O}_{\mathcal{E}} \cong \mathcal{E} \times \mathbb{C} \xrightarrow{\pi_2} \mathbb{C}.$$

We get a fibration with $\mathcal{Y}_t := \pi^{-1}(t)$, which is smooth for $t \neq 0$ and $\mathcal{Y}_0 = Y \cup \mathcal{E}$. The intersection $E = Y \cap \mathcal{E}$ restricts to the degree $(1,1)$ hypersurface over each fiber of $\mathcal{E} \to S$. Indeed in each fiber the equation for π in coordinates reads as

$$t = \sum_{i=0}^{r} x_i y_i.$$

¿From this, we also have that $\mathcal{Y}_t \cong \mathcal{E} \backslash E$ for all $t \neq 0$ under the projection p.

Let $\mathcal{X}_t$, $\mathcal{X}'_t$ and $\bar{\mathcal{X}}_t$ be the proper transforms of $\mathcal{Y}_t$ in $\mathcal{X}$, $\mathcal{X}'$ and $\bar{\mathcal{X}}$. For $t \neq 0$, all maps in the diagram

$$\begin{tikzcd} & \mathcal{Y}_t \arrow[dl] \arrow[dr] & \\ \mathcal{X}_t \arrow[dr] & & \mathcal{X}'_t \arrow[dl] \\ & \bar{\mathcal{X}}_t & \end{tikzcd}$$

are isomorphisms. For $t = 0$ this is the Mukai flop. Thus local Mukai flops are limits of isomorphisms. More precisely, we have

THEOREM 6.2. *There is a projective compactification $\widehat{\mathcal{Y}} \to \mathbb{P}^1$ which deforms the projectivized local model of Mukai flop*

$$\widehat{\mathcal{X}}_0 = \mathbb{P}_Z(T^*_{Z/S} \oplus \mathcal{O}) \dashrightarrow \mathbb{P}_{Z'}(T^*_{Z'/S} \oplus \mathcal{O}) = \widehat{\mathcal{X}}'_0$$

into isomorphisms $\widehat{\mathcal{X}}_t \cong \widehat{\mathcal{X}}'_t \cong \mathcal{E}$ for all $t \neq 0$.

Moreover, $\widehat{\mathcal{Y}} \to \mathbb{P}^1$ is the blow-up of $\mathcal{E} \times \mathbb{P}^1$ along $E \times \{0\}$, the degeneration to normal cone of the pair $(\mathcal{E}, E)$ with E being the relative $(1,1)$ divisor of

$$\mathcal{E} = \mathbb{P}_S(F) \times_S \mathbb{P}_S(F^*)$$

over S. $\widehat{\mathcal{X}}_0$, $\widehat{\mathcal{X}}'_0$ and $\widehat{\widehat{\mathcal{X}}}_0$ are the contractions of $\mathcal{E} \subset \widehat{\mathcal{Y}}_0$ along the two rulings and the double ruling respectively.

Proof. We first consider the compactified normal bundle

$$\bar{\mathcal{Y}} = \mathbb{P}_{\mathcal{E}}(\mathcal{O}(-1,-1) \oplus \mathcal{O}) \dashrightarrow \mathbb{P}^1$$

which extends the map π by sending the infinity divisor $\mathcal{E}_\infty \cong \mathcal{E}$ to $\infty \in \mathbb{P}^1$.

It is clear that $E_\infty := \bar{Y} \cap \mathcal{E}_\infty$ is the "*axis*" where π is not defined. Indeed E_∞ is the boundary divisor of every $\mathcal{Y}_t$. Thus the blow-up

$$\begin{tikzcd} \widehat{\mathcal{Y}} := \mathrm{Bl}_{E_\infty}\bar{\mathcal{Y}} \arrow[d] \arrow[dr, "\hat{\pi}"] & \\ \bar{\mathcal{Y}} \arrow[r, dashed, "\pi"] & \mathbb{P}^1 \end{tikzcd}$$

resolves the indeterminacy to get a morphism $\hat{\pi} : \widehat{\mathcal{Y}} \to \mathbb{P}^1$. $\widehat{\mathcal{Y}}_t$ is the compactification of $\mathcal{Y}_t$ by adding E at infinity, hence $\widehat{\mathcal{Y}}_t \cong \mathcal{E}$ for all $t \neq 0$.

We then have a compactified diagram as expected:

$$\begin{tikzcd} & \widehat{\mathcal{Y}}_t \arrow[dl] \arrow[dr] & \\ \widehat{\mathcal{X}}_t \arrow[dr] & & \widehat{\mathcal{X}}'_t \arrow[dl] \\ & \widehat{\widehat{\mathcal{X}}}_t & \end{tikzcd}$$

For $t = 0$, by the very construction of Mukai flops from the ordinary flops, we have $\mathbb{P}_Z(T^*_{Z/X}) \cong E$. So the compactification $\widehat{\mathcal{X}}_0$ coincides with $\mathbb{P}_Z(T^*_{Z/S} \oplus \mathcal{O})$. Similarly $\widehat{\mathcal{X}}'_0 \cong \mathbb{P}_{Z'}(T^*_{Z'/S} \oplus \mathcal{O})$.

For the second statement, again by our construction $\widehat{\mathcal{Y}}_0 = \mathcal{E} \cup \widehat{Y}$ with $\widehat{Y}$ the total space of the $\mathbb{P}^1$ bundle

$$\mathbb{P}_E(\mathcal{O}_E(-1,-1) \oplus \mathcal{O}) \cong \mathbb{P}_E(\mathcal{O} \oplus \mathcal{O}_E(1,1)).$$

This is precisely the exceptional divisor coming from the blow-up

$$\widetilde{\mathcal{Y}} = \mathrm{Bl}_{E\times\{0\}}\mathcal{E} \times \mathbb{P}^1.$$

The theorem follows from an easy comparison of $\widetilde{\mathcal{Y}}$ with $\widehat{\mathcal{Y}}$. $\square$

In particular all interesting invariants which are continuous under deformations are preserved. For example, the diffeomorphism type, Hodge type and quantum cohomology rings etc.. To be more precise, since the fiber product satisfies the base change property and for ordinary flops the fiber product equals the graph closure, the canonical isomorphism of Chow motives of projective local models of Mukai flops $f : X \dashrightarrow X'$ is clearly to be induced by the correspondence $[X \times_{\bar{X}} X']$, which is the $t = 0$ fiber of the graph of $\mathcal{X} \dashrightarrow \mathcal{X}'$:

$$\mathcal{F} := [X \times_{\bar{X}} X'] = [\bar{\Gamma}_f] + [\mathcal{E}] \in A^*(X \times X')$$

where $[\bar{\Gamma}_f] \equiv Y := \mathrm{Bl}_Z X = \mathrm{Bl}_{Z'} X'$. For global (twisted) Mukai flops we also consider the fiber product as the proposed correspondence $\mathcal{F}$.

The quantum cohomologies are not just isomorphic, in fact all quantum corrections attached to the extremal ray are zero: If not, then the deformation invariance of Gromov-Witten invariants implies that some extremal curve class $d\ell \in NE(X)$ survives as an effective curve in a nearby fiber as $C_t \subset \mathcal{X}_t \cong \mathcal{X}'_t$, then the class

$$[C'_t] = \mathcal{F}_t[C_t] \sim \mathcal{F}d\ell = -d\ell'.$$

is both effective and anti-effective on $\mathcal{X}'$, which is a contradiction. (For simple Mukai flops, the invariants on $d\ell$ are zero have also been proved by Hu and Zhang [8] by direct computation via localizations.)

For a global Mukai flop, the local deformation equivalence may fail to extend to a global deformation equivalence since there are in general obstructions to extend deformations from local to global. (For hyper-Kähler manifolds or more generally Calabi-Yau manifolds such global deformations do exists.) Nevertheless, together with the degeneration analysis, the local deformation equivalence do lead to global results:

Theorem 6.3. *For any Mukai flop $f : X \dashrightarrow X'$ (not necessarily being simple), X is diffeomorphic to X' and both have isomorphic Chow motives, Hodge structures and full Gromov–Witten theory (in all genera) under the correspondence $\mathcal{F}$. Moreover, all quantum corrections attached to the extremal ray vanish.*

Proof. The diffeomorphism is obtained by patching the local deformation equivalence and the identity map on $X\backslash Z \cong X'\backslash Z'$.

For Chow motives, we investigate the induced mapping on Chow groups as in §2. For any T, $\mathrm{id}_T \times f : T \times X \dashrightarrow T \times X'$ is also a Mukai flop, with base

S being replaced by $T \times S$. Since the correspondence $\mathcal{F}$ is compatible with base change, to prove that $\mathcal{F}^* \circ \mathcal{F} = \Delta_X$, by the identity principle, we only need to show that $\mathcal{F}^*\mathcal{F} = \mathrm{id}$ on $A^*(X)$ for any Mukai flop. Let $p : X \times X' \to X$ and $p' : X \times X' \to X'$ be the projections. From

$$\mathcal{F}W = p'_*(([\bar{\Gamma}_f] + [\mathcal{E}]).p^*W)$$

and the property of intersection product we see that the $\mathcal{F}^* \circ \mathcal{F} = \Delta_X$ is really a local statement which depends only on the normal bundles $N_{Z/X}$ and $N_{Z'/X}$. Thus the identity follows from the case of local models. Similarly $\mathcal{F} \circ \mathcal{F}^* = \Delta_{X'}$. So $\mathcal{F}$ induces an isomorphism on Chow motives of X and X'. The Hodge realizations leads to equivalence of Hodge structures.

Now we treat the Gromov-Witten invariants. As in the case of ordinary flops, we consider degeneration to normal cone $W \to \mathbb{A}^1$ of X and $W' \to \mathbb{A}^1$ of X' respectively. $W_0 = Y \cup X_{\rm loc}$ with $Y = \mathrm{Bl}_Z X$ and $X_{\rm loc} = \mathbb{P}_Z(T^*_{Z/S} \oplus \mathcal{O})$. Similarly $W'_0 = Y' \cup X'_{\rm loc}$ with $Y' = \mathrm{Bl}_{Z'} X'$ and $X'_{\rm loc} = \mathbb{P}_{Z'}(T^*_{Z'/S} \oplus \mathcal{O})$. By definition $Y = Y'$ and we have the induced Mukai flop for local models $f : X_{\rm loc} \dashrightarrow X'_{\rm loc}$.

By the degeneration formula, any Gromov-Witten invariant $\langle \alpha \rangle^X_{g,n,\beta}$ splits into sum of products of relative invariants of (Y, E) and $(X_{\rm loc}, E)$. Now we compare it with the similar splitting of $\langle \mathcal{F}\alpha \rangle^X_{g,n,\beta}$ into (Y, E) and $(X'_{\rm loc}, E)$.

Notice that most of the setting on degeneration analysis in §4 is still valid in the Mukai case. In particular, the cohomology reduction (Proposition 4.4) works in the Mukai case too.

In fact, the situation now is very simple. We match the relative invariants on (Y, E) from both sides and then we need to compare only the cases $(X_{\rm loc}, E)$ and $(X'_{\rm loc}, E)$. But they are deformation equivalent while the deformations leave the boundary divisor E unchanged. By the deformation invariance of (relative) Gromov-Witten theory and the fact that $\mathcal{F}$ is induced from this deformation, we find that the relative invariants are also the same on this part. Hence we have proved

$$\langle \alpha \rangle^X_{g,n,\beta} = \langle \mathcal{F}\alpha \rangle^{X'}_{g,n,\mathcal{F}\beta}$$

for any g, n, β including descendent invariants. Namely the full GW theory on X and X' are equivalent.

The statement on vanishing of GW invariants of extremal rays follows from the previous discussion. The proof is now complete. □

REMARK 6.4. *Instead of using deformation invariance of relative GW theory, we may also proceed in the same way as the case of ordinary flops, at least for simple Mukai flops. By Proposition 4.6, the equivalence problem is reduced to the case of absolute invariants and then we may use the deformation invariance of absolute GW theory to conclude. Indeed the deformation* *invariance of relative GW theory can be deduced form the absolute case and the result in [20].*

Remark 6.5. *For twisted Mukai flops we take $F' = F^* \otimes L$ and $\eta_S := L$. The pairing $F \times_S F' \to \eta_S$ is simply $F \times_S (F^* \otimes L) \to L$. Since*

$$\bar{\phi}'^* \mathcal{O}_{\mathbb{P}(F^* \otimes L)}(-1) = \bar{\phi}'^* (\mathcal{O}_{\mathbb{P}(F^*)}(-1) \otimes \bar{\psi}'^* L) = \bar{\phi}'^* \mathcal{O}_{\mathbb{P}(F^*)}(-1) \otimes \bar{g}^* L,$$

the linear map $\mathcal{Y} \to \bar{g}^* L$ is obtained by tensoring the corresponding map for Mukai flops with $\bar{g}^* L$. The inverse image of the zero section gives $Y \cup \mathcal{E}$. Again the proper transforms of Y in various spaces give rise to the twisted Mukai flop. The difference is that since $\bar{g}^* L$ is not a trivial bundle, we do not have a fibration structure $\mathcal{Y} \to \mathbb{C}$ as before. But we still get the equivalence of Chow motives via the fiber product.

Example 6.6. *To see how the extra component corrects the graph closure, we shall carry out the detailed computations for the case of simple Mukai flops. So $Z \cong \mathbb{P}^r$, $N_{Z/X} = T_Z^*$ and $E \subset \mathbb{P}^r \times \mathbb{P}^r$ is the universal family of lines in $\mathbb{P}^r$ from both sides, namely, it is the hypersurface of bi-degree $(1,1)$. By weak Lefschetz, $H^2(E) = \operatorname{Pic} E = \mathbb{Z}x|_E \oplus \mathbb{Z}y|_E$ with x and y the generators of $\operatorname{Pic} \mathbb{P}^r \times \mathbb{P}^r$ as pull backs of h and h'. As in the ordinary case, $N_{E/Y} = \mathcal{O}_E(-1,-1) := \bar{\phi}^* \mathcal{O}_Z(-1) \otimes \bar{\phi}'^* \mathcal{O}_{Z'}(-1)$.*

Let $\mathcal{F}_0 = [\bar{\Gamma}_f]$. The argument to compute $\mathcal{F}_0$ as in the ordinary case fails precisely when $\alpha \in A^r(X)$, so we would like to find $\mathcal{F}_0[Z]$. Since $\phi^*[Z] = j_*(c_{r-1}(\mathcal{E}))$, with $\mathcal{E}$ defined by $0 \to N_{E/Y} \to \bar{\phi}^* N_{Z/X} \to \mathcal{E} \to 0$, we get

$$\begin{aligned} c_{r-1}(\mathcal{E}) &= \big((1-x)^{r+1}(1-(x+y))^{-1}|_E\big)_{(r-1)} \\ &= \big((x+y)^{r-1} - C_1^{r+1} x (x+y)^{r-2} + \cdots + (-1)^{r-1} C_{r-1}^{r+1} x^{r-1}\big)\big|_E \\ &= (y^{r-1} - 2y^{r-2}x + 3y^{r-3}x^2 + \cdots + (-1)^{r-1} r x^{r-1})|_E. \end{aligned}$$

So

$$\mathcal{F}_0[Z] = \phi'_* \phi^*[Z] = (-1)^{r-1} r [Z'],$$

which implies that $\mathcal{F}_0$ induces isomorphism on cohomologies over $\mathbb{Q}$, but not over $\mathbb{Z}$.

For $0 < s \le r$, since $E \sim x + y$, we have

$$\begin{aligned} \phi^* h^s &= j_*(c_{r-1}(\mathcal{E}).\bar{\phi}^* h^s) \\ &= (y^{r-1} - 2y^{r-2}x + 3y^{r-3}x^2 + \cdots + (-1)^{r-1} r x^{r-1})|_E . x^s \\ &= (y^r - y^{r-1}H + \cdots + (-1)^{r-1} y x^{r-1} + (-1)^r (1-r) x^r) x^s \\ &= x^s y^r - x^{s+1} y^{r-1} + \cdots + (-1)^{r-s} x^r y^s. \end{aligned}$$

By symmetry this implies that $\mathcal{F}_0(h^s) = (-1)^{r-s} h'^s$ when $s \ne 0$.

Let $\mathcal{F} = [X \times_{\bar{X}} X'] = \mathcal{F}_0 + \mathcal{F}_1$ with $\mathcal{F}_1 = Z \times_S Z' = [\mathbb{P}^r \times \mathbb{P}^r]$. We claim that $\mathcal{F}_1[Z] = (-1)^r(r+1)[Z']$ and $\mathcal{F}_1 h^s = 0$ for $s \neq 0$. Indeed,

$$\mathcal{F}_1[Z] = p'_*(p^{-1}[Z].[Z \times Z'])$$

with p (resp. p') the projection of $X \times X'$ to X (resp. X'). Then

$$Z^2 = c_r(N_{Z/X}) = c_r(T_Z^*) = (-1)^r\chi(\mathbb{P}^r) = (-1)^r(r+1).$$

So $\mathcal{F}_1[Z] = p'_*([Z \times X'].[Z \times Z']) = (-1)^r(r+1)[Z']$. For $\mathcal{F}_1 h^s$, notice that we may choose $W \sim Z$ with $W \cap h^s = \emptyset$. Hence $\mathcal{F}_1 h^s = p'_*([h^s \times X'].[W \times Z']) = 0$.

Thus $\mathcal{F}(h^s) = (-1)^{r-s} h'^s$ for $0 \leq s \leq r$ and $\mathcal{F}$ induces integral isomorphisms.

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