Algebraic surfaces

Birational maps on surfaces and minimal models

Remark 0.1. *Most of the material of this section can be found in [Beauville]. The last example is called the elementary transform of ruled surface. It can be found in [Ha, V.5, p.416]*

We are going to study birational maps on surfaces and introduce the notion of *minimal models* in this section.

Recall that by a rational map $f : X \dashrightarrow Y$ we mean an equivalent classes (U, f) of morphism $f : U \to Y$. Roughly speaking, a rational map is a map not everywhere defined. Sometimes the map can be extended to a larger domain. For example, let $(U, f) \sim (V, g)$ be equivalent rational maps, i.e. $f = g$on $U \cap V$, then one can define a map on $U \cup V$. Indeed, one can extend (U, f) to $U_0 := \cup_{V \in \mathcal{C}} V$, where $\mathcal{C}$ denotes the equivalent class. The point in $X - U_0$ is called *points of indeterminacy.*

Proposition 0.2. *Let $f : X \dashrightarrow Y$ be a rational map to a projective variety Y. Then point of indeterminacy $X - U_0$ has codimension ≥ 2.*

In particular, if $\dim X = 2$ *then the set of points of indeterminacy is finite.*

Proof. Since Y is projective, we may assume that $Y = \mathbb{P}^n$. Let H be a hyperplane in $\mathbb{P}^n$ and $D = f^*H$. Let $Z_0, .., Z_n$ be the homogeneous coordinates of $\mathbb{P}^n$. Then $div(Z_i \circ f)$ gives a divisor $F_i \in |D|$.

The point of indeterminacy are exactly the common zero of $Z_i \circ f$. (Note that $Z_i \circ f$ is a section in $H^0(X, \mathcal{O}(D))$, which is locally regular. Thus we only need to worry about common zeros)

If $W_1 \subset X$ is a codimension 1 subvariety of point of indeterminacy. Then W_1 is the common zero hence $W_1 < F_i$ for all i. Let $D_1 = D - W_1$ and $s \in H^0(X, \mathcal{O}(W_1))$ a section defining W_1, i.e. $div(s) = W_1$. One sees that $\frac{Z_i \circ f}{s} \in H^0(X, \mathcal{O}(D_1))$ which are locally regular.

We consider the map $f_1 : X \dashrightarrow \mathbb{P}^n$ by $[\frac{Z_0 \circ f}{s}, ..., \frac{Z_n \circ f}{s}]$. It's clear that $f_1 = f$ on U_0. (Potentially, f_1 might have eliminated indeterminacy on W_1). Thus f_1 can be extended to a larger defining domain.

By continuing this process, we get $f_1, f_2, ...$ associated to effective divisors $D_1 \gneqq D_2 \gneqq D_3 ....$ Since effective divisor can have only finitely many non-zero places, this process must terminate. That is, we reach $f_n : X \to \mathbb{P}^n$ without point of indeterminacy of codimension 1. $\square$

Indeed, we can eliminate those finite points of indeterminacy by blowing-ups.

Theorem 0.3 (Elimination of indeterminacy)**.** *Let $f : X \dashrightarrow Y$ be a rational map from a surface to a projective variety Y. Then there exists a morphism $p : X' \to X$ which is a composition of blowing-ups, together with a morphism $f' : X' \to Y$ such that $f' \sim f \circ p$.*

Proof. Since Y is projective, we may assume that $Y = \mathbb{P}^n$. Let H be a hyperplane in $\mathbb{P}^n$ and $D = f^*H$. Let $Z_0, .., Z_n$ be the homogeneous coordinates of $\mathbb{P}^n$. Then $div(Z_i \circ f)$ gives a divisor $F_i \in |D|$.

The point of indeterminacy are exactly the common zero of $Z_i \circ f$. Suppose that $x \in X$ is a point of indeterminacy. We consider $\pi : X_1 = Bl_x(X) \to X$. By composition, one has a map

$$f_1 : X_1 \to X \dashrightarrow \mathbb{P}^n$$

given by $[Z_0 \circ f \circ \pi, ..., Z_n \circ f \circ \pi]$. Note that $div(Z_i \circ f \circ \pi)$ now gives divisors in $|\pi^* D|$.

Recall that for each i, $div(Z_i \circ f)$ passes through x of multiplicity m_i. Let $m = \min_{i=0,...,n} m_i$. It thus follows that $div(Z_i \circ f \circ \pi) > mE$ for each i. Let $s \in H^0(X', \mathcal{O}_{X'}(E)$ be the section defining E, i.e. $div(s) = E$. We then consider the map (as in the previous Proposition)

$$f_1' : X_1 \dashrightarrow \mathbb{P}^n,$$

by $[\frac{Z_0 \circ f \circ \pi}{s^m}, ..., \frac{Z_n \circ f \circ \pi}{s^m}]$. One sees that f_1' extends f_1 and f_1' is defined on all but finite point on E.

If there is point of indeterminacy for f_1, we then continue this process to obtain $f_k : X_k \dashrightarrow \mathbb{P}^n$ inductively. It remains to show that this process must stop.

Notice that we may assume that $f : X \dashrightarrow \mathbb{P}^n$ is non-constant and non-degenerate. Thus pick any two general hyperplane H_i, H_2 in $\mathbb{P}^n$, one has $H_1.H_2.f(X) \geq 0$. Thus $f^*H_1.f^*H_2 = D^2 \geq 0$.

Notice that the divisor corresponds to f_1 is $D_1 := \pi^*D - mE$, one has $D_1^2 = D^2 - m^2 \geq 0$. By applying this observation to all $f_i : X_i \to \mathbb{P}^n$. One has

$$D^2 \gneq D_1^2 \gneq D_2^2 ... \geq 0.$$

Hence it must stop at some D_k, thus one has that $f_k : X_k \to \mathbb{P}^n$ has no point of indeterminacy. Set $X' := X_k, f' := f_k$ then we are done. □

The following property is crucial in the study of birational map of surfaces.

Proposition 0.4. *Let $f : X \to Y$ be a birational morphism. If $y \in Y$ is a point of indeterminacy of f^{-1}, then f factors through $\pi : Bl_y(Y) \to Y$. That is, there is a morphism $f' : X \to Bl_y(Y)$ such that $f = \pi \circ f'$.*

Proof. The proof is pretty long so that we will not include it here. Please see [Beauville] for the detail. □

Corollary 0.5. *Let $f : X \to Y$ be a birational morphism, then there is $\pi_k : Y_k \to Y$ which is composition of blowing-ups and an isomorphism $\epsilon : X \to Y_k$ such that $f = \pi_k \circ \epsilon$.*

Proof. If f is an isomorphism then nothing to prove. It f is not an isomorphism, then there must be a point $y \in Y$ such that f^{-1} is undefined at y. One has $X \to Y_1 := Bl_y(Y) \to Y$. One can continue this process unless we have an isomorphism.

It remains to show that this process must terminate. We need to find an invariant to control the termination. A naive approach is trying to count points of indeterminacy at each step. However, this does not behave well because from $Y \dashrightarrow X$ to $Y_1 \dashrightarrow X$, we eliminate the undefining point y but there might have some more point of indeterminacy on $E \subset Y_1$. Thus we need a more refined invariant.

We consider the rank of Neron-Severi group. Recall that the Neron-Severi group is the *algebraic equivalent classes of divisors*. It seems difficult to understand what it is. But anyway, it's an finitely generated abelian group. Moreover,

$$NS(Bl_x(X)) = NS(X) \oplus \mathbb{Z}[E].$$

In particular,

$$rk(NS(Bl_x(X)) = rk(NS(X)) + 1.$$

(Remark: If X is defined over $\mathbb{C}$, then $NS(X) = im(H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}))$ which is of course of finite rank).

Now one has

$$rk(NS(X))... \gneq rk(NS(Y_2)) \gneq rk(NS(Y_1)) \gneq rk(NS(Y)).$$

It's clear that the process of producing $Y_1, Y_2 ...$ must terminate at Y_k for some k since $rk(NS(X))$ is finite. Hence one has $X \cong Y_k$ cause otherwise one can produce Y_{k+1}. This completes the proof. □

The corollary says that a birational morphism of surfaces is basically composition of blowing-ups and isomorphism. Together with the theorem on elimination of indeterminacy, we have the following:

Corollary 0.6. *Let $f : X \dashrightarrow Y$ be a birational map of surfaces. Then there is a surface Z and morphisms $g : Z \to X$, $h : Z \to Y$ such that $h \sim f \circ g$, where g, h are composition of blowing-ups and isomorphisms.*

Let X be a smooth surface, we can consider $Bir(X)$ to be the birational equivalent class of smooth surfaces which are birational to X. We have seen that any two surfaces in $Bir(X)$ are connected by blowing-ups and isomorphisms.

In what follows, we would like to consider $Bir(X)^0$ to be the birational equivalent class modulo isomorphism. Then there is a natural partial ordering on $Bir(X)^0$ by $[X_1] \geq [X_2]$ if there is a birational morphism $f : X_1 \to X_2$, where $[X_1]$ denotes the isomorphic class of X_1. We have seen that $[Bl_x(X)] \gneq [X]$ and if $[X] \geq [Y]$ then $[X] = [Y_k]$ for some composition of blowing up $Y_k \to Y$.

Our next goal is to show that there exist a minimal element in $Bir(X)^0$, which we call it *a minimal model of* X.

Definition 0.7. *A non-singular surface X is minimal if for any morphism $f : X \to Y$ to a non-singular surface, f is an isomorphism.*

(i.e. if $[X] \geq [Y]$, then $[X] = [Y]$ in $Bir(X)^0$).

Theorem 0.8. *Let X be a non-singular surface, then there exist a minimal surface X' together with a birational morphism $f : X \to X'$. In other words, minimal model exists.*

Proof. If X is not minimal, then there is an surface Y and a birational morphism $f : X \to Y$. Since f is composition of isomorphism and blowing-ups. We may assume that there is an X_1 and $X \cong Bl(X_1)$. If X_1 is minimal then we are done, otherwise, one has X_2 and $X_1 \cong Bl(X_2)$ similarly. Thus one has sequence of surfaces

$$X \to X_1 \to X_2...$$

However, $rk(NS(X_{i+1})) = rk(NS(X_i) - 1$. Thus the sequence must stop at a minimal model. □

An convenience way to check minimality for surface is the following:

Theorem 0.9. *Let X be a non-singular surface. then X is minimal if and only if X has no (-1)-curves.*

Proof. If X has an (-1)-curve, then by CAstelnuovo's contraction theorem, there is a contraction $X \to X'$ contracting the (-1)-curve. Hence X is not minimal.

On the other hand, if X is not minimal, then as we have seen above, $X \cong Bl(X_1)$ for some X_1. In particular, there the exceptional divisor is an (-1)-curve. □

However, minimal model is not always unique.

Example 0.10. *Let $X = C \times \mathbb{P}^1$, where C is a curve of genus ≥ 2. X is a ruled surface by considering $\pi : X \to C$.*

Recall that by a ruled surface, we mean a surface X together with a morphism $\pi : X \to B$ to a curve B such that each fiber $F_b := \pi^{-1}(b) \cong \mathbb{P}^1$.

Fix now a point $x \in X$ lying over $b \in C$. We consider $Z = Bl_x(X)$. And there is a composition map $\pi_Z : Z \to C$. Now over $b \in C$, $\pi_Z^{-1}(b) = \tilde{F}_b + E$. Easy computation show that $\tilde{F}_b$ is a (-1)-curve on Z. One can contract $\tilde{F}_b$ and obtained a surface Y. There is a $\pi_Y : Y \to C$. But one can prove that $Y \ncong C \times \mathbb{P}^1 = X$.

However, both X and Y are minimal model of Z. Hence minimal model is not unique.