Algebraic surfaces

Sheaf Cohomology

Remark 0.1. *For definition and basic properties of sheaves, see [Ha, II.1]. For more detail on derived functors, see [Ha, III.1]. And the construction of Čech cohomology can be found in [Ha, III. 4]. The definition of sheaf of module, quasi-coherent, coherent sheaves can be found in [Ha, II. 5].*

It was in 50's that sheaves was introduced into algebraic geometry. A monumental work is Serre's *faisceaux algebriques coherents*. Now it turns out to be the one of the basic tool in algebraic geometry.

The idea of sheaves is that if one would like to study functions on a variety X, it's useful to study function on all open sets. Once local functions can be patched together, then one get the required global ones.

To formulate this idea, we first define *presheaf*. And we consider a sheaf as a presheaf which allow one to patch local datum to get global datum.

Definition 0.2. *Let X be a topological space. A presheaf $\mathcal{F}$ (of abelian groups) is an assignment from open sets of X to abelian groups and if $V \subset U$, there is a homomorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that:*

(1) $\mathcal{F}(\emptyset) = \{0\}$.
(2) $\rho_{UU} = id_U$ *for any* U.
(3) *if* $W \subset V \subset U$, *then* $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

A presheaf $\mathcal{F}$ is a sheaf if it satisfies the following properties:

(1) *if $U = \cup_i U_i$, and there is a $s \in \mathcal{F}(U)$ such that $\rho_{UU_i}(s) = 0 \in \mathcal{F}(U_i)$ for all i, then $s = 0 \in \mathcal{F}(U)$.*
(2) *if $U = \cup_i U_i$, and there are $s_i \in \mathcal{F}(U_i)$ such that $\rho_{U_i,U_{ij}}(s_i) = \rho_{U_j,U_{ij}}(s_j) \in \mathcal{F}(U_i \cap U_j)$ for all i, j, then there is an $s \in \mathcal{F}(U)$ such that $\rho_{UU_i}(s) = s_i \in \mathcal{F}(U_i)$ for all i.*

Most of the natural sheaves are coming from various consideration of functions.

Example 0.3. (1) *Let X be a topological space. One has the sheaves of real-valued (or complex-valued) continuous function.*
(2) *By giving $\mathbb{Z}$ the discrete topology, one can even consider the sheaf of integral-valued continuous function. (This is the sheaf of locally constant function with value in $\mathbb{Z}$. Note the constant presheaf given by $\mathcal{F}(U) = \mathbb{Z}$ is not a sheaf if X is not connected.*
(3) *Let X be a variety. We consider $\mathcal{O}_X$ to be the presheaf such that $\mathcal{O}_X(U) = \{$regular functions on $U\}$. Then $\mathcal{O}_X$ is a sheaf.*

Definition 0.4. *Let $\mathcal{F}, \mathcal{G}$ be presheaves. By a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ we mean a collection of group homomorphism $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that it is compatible with the restriction maps ρ_{UV}*

Remark 0.5. *Given a morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$. One has naturally presheaves $ker(\varphi), im(\varphi), coker(\varphi)$. However, if both $\mathcal{F}, \mathcal{G}$ are sheaves, then $im(\varphi), coker(\varphi)$ are not necessarily sheaves. Nevertheless, one can define the sheafification of a presheaf. Therefore, when we say the sheaf $im(\varphi)$ (resp. $coker(\varphi)$), we really mean the sheafification of it.*

Example 0.6. *Let X be a variety and $Z \subset X$ be a subvariety. Let*

$$\mathcal{O}_{Z,X}(U) := \{f_{|Z} \mid f \in \mathcal{O}_X(U), f_{|Z} \text{ is regular}\},$$

$$\mathcal{I}_Z(U) := \{f \in \mathcal{O}_X(U) | f \text{ vanishes along } Z\}.$$

Then both $\mathcal{O}_{Z,X}$ and $\mathcal{I}_Z$ are sheaves. Moreover, one have exact sequences

$$0 \to \mathcal{I}_Z(U) \to \mathcal{O}_X(U) \to \mathcal{O}_{Z,X}(U),$$

for all U.

Remark 0.7. *A convenient and important notion if the stalk of a presheaf (resp. sheaf) at a given point. The stalk of $\mathcal{F}$ at x, denoted $\mathcal{F}_x$, is defined as the direct limit of germs (U, f), i.e. $\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U)$.*

Given a sequence of sheaves $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$, it's tempting to ask if the kernel of $\psi : \mathcal{G} \to \mathcal{H}$ equal to the image of $\varphi : \mathcal{F} \to \mathcal{G}$. It's not easy to check this directly since we need to sheafify the presheaf $im(\varphi)$ before one can really compare them. However, since sheafification preserve stalks. It turns out we have the the following criterion:

A sequence of sheaves $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact if and only if $\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$ is exact for all $x \in X$.

Exercise 0.8. *Check that the sequences*

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_{Z,X} \to 0$$

is exact.

Example 0.9. *Let $D \in Div(X)$ be a divisor. One can have a sheaf $\mathcal{O}_X(D)$ such that*

$$\mathcal{O}_X(D)(U) := \{f \in K(X) | div(f) + D \geq 0 \text{ on } U\}.$$

It's clear that the global sections $\Gamma(X, \mathcal{O}(D)) = \mathcal{O}_X(D)(X) = \mathcal{L}$.

Note that if $D_2 \geq D_1$, then one has $\mathcal{O}(D_1) \hookrightarrow \mathcal{O}(D_2)$.

Moreover, if $Z \subset X$ is a codimension 1 subvariety, then $\mathcal{I}_Z = \mathcal{O}(Z)$.

Considering the category of sheaves (of abelian groups) on X, one has the functor $\Gamma(X, \cdot)$ to the category of abelian group. This functor is left exact, i.e. if we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0,$$

then

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H})$$

is exact. (But $\Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H})$ maybe not surjective). One can construct the right derived functor $R^i\Gamma(X, \cdot)$ to complete the sequnce

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H}) \to$$

$$R^1\Gamma(X, \mathcal{F}) \to R^1\Gamma(X, \mathcal{G}) \to R^1\Gamma(X, \mathcal{H}) \to R^2\Gamma(X, \mathcal{F}) \to ...$$

We denote $H^i(X, \mathcal{F}) := R^i\Gamma(X, \mathcal{F})$, where $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

Remark 0.10. *The construction is as following: Take the (injective) resolution of sheaf. Then the exact sequence of sheaves gives an exact sequence of complexes of (injective) sheaves. The required right derived functor can be obtained via the global sections of these injective sheaves.*

Another construction is the Čech cohomology.

Before we move on, we recall a basic facts on dimension of vector space.

Exercise 0.11. *Let $\phi : V \to W$ be a linear transformation between V, W. Then one has*

$$\dim im(\phi) + \dim ker(\phi) = \dim V.$$

In particular, if one has an exact sequence of vector spaces

$$0 \to V_1 \to V_2 \to V_3 \to 0,$$

then $\dim V_2 = \dim V_1 + \dim V_3$.

Let

$$0 \to V_1 \to V_2 \to ... \to V_n \to 0$$

be an exact sequence of vector spaces. Show that

$$\sum (-1)^i \mathrm{dim} V_i = 0.$$

Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ *be an exact sequence of sheaves. Suppose that* $H^i(X, \cdot) = 0$ *for* $i > n$*. We define*

$$\chi(X, \cdot) := \sum (-1)^i \mathrm{dim} H^i(X, \cdot).$$

Show that $\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H})$*.*

We are now ready to look at Riemann-Roch theorem via the language of sheaf cohomology.

Theorem 0.12 (Serre)**.** *Let* X *be a variety of dimension* n *and* $\mathcal{F}$ *is a coherent sheaf on* X*, then* $H^i(X, \mathcal{F}) = 0$ *for* $i > n$*.*

First consider the case that $\mathrm{dim} X = 1$. For a divisor D and a point $P \in X$, one has the exact sequence

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D + P) \to k(P) \to 0,$$

where $k(P)$ denotes the constant sheaf k at P. One then has

$$\chi(D + P) = \chi(D) + 1.$$

Write $D = D_+ - D_-$ into positive and negative parts, by working on χ Inductively, one has $\chi(D) = \chi(D_+) + deg(D_-)$ and $\chi(D_+) = \chi(0) + deg(D_+)$. It follows that

$$\chi(D) = \chi(0) + deg(D).$$

Theorem 0.13 (Serre duality)**.** *Let* $\mathcal{F}$ *be a coherent sheaf on a non-singular* n*-dimensional projective variety* X*. Then* $H^i(X, \mathcal{F}) \cong H^{n-i}(X, \omega_X \otimes \mathcal{F}^\vee)^\vee$*. Where* ω_X *is the dualizing sheaf (which is the canonical sheaf if* X *is non-singular).*

In particular, $h^i(X, \mathcal{F}) := \mathrm{dim} H^i(X, \mathcal{F}) = h^{n-i}(X, \omega_X \otimes \mathcal{F}^\vee)$*.*

Remark 0.14. *This actually works in a more general setting. Please see [Ha, III.7].*

Turning back to 1-dimensional case. By Serre duality,

$$\chi(X, D) := h^0(X, \mathcal{O}(D)) - h^0(X, \mathcal{O}(D))$$
$$= h^0(X, \mathcal{O}(D) - h^0(X, \omega \otimes \mathcal{O}(-D)) = \mathrm{dim} \mathcal{L}(D) - \mathrm{dim} \mathcal{L}(K_X - D).$$

And similarly,

$$\chi(X, 0) := h^0(X, \mathcal{O}) - h^1(X, \mathcal{O}) = 1 - g(X).$$

Thus we have the Riemann-Roch theorem for 1-dimensional X.

We can now try to look at Riemann-Roch theorem on surfaces. Let P be a prime divisor (an irreducible subvariety of codimension 1) on a surface X and D is a arbitrary divisor. We have

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D + P) \to \mathcal{O}_P(D + P) \to 0.$$

The point is what is $\chi(P, \mathcal{O}_P(D+P))$? There are some potential problems: The first one is that P maybe singular, so we don't have Riemann-Roch on P immediately. The second one is, even though P is non-singular, what's the genus of P? And finally, what's the degree of the divisor $D + P|_P$?

We need to work a little bit harder to get Riemann-Roch theorem on surfaces.