Algebraic surfaces

Divisors

Remark 0.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with residue field $k := A/\mathfrak{m}$. We said that A is a regular local ring if $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$. A ring is regular if the localization at every maximal ideal is a regular local ring.*

Remark 0.2. *Let Y be an affine variety with coordinate ring $A(Y)$. And $P \in Y$ is a point. We say Y is non-singular at P if $\mathcal{O}_{P,Y}$ is a regular local ring. We say that Y is non-singular if it's non-singular at every point.*

We say that Y is normal at P if $\mathcal{O}_{P,Y}$ is a integrally closed. We say that Y is normal if it's non-singular at every point.

In general, one has "non-singular $\Rightarrow$ normal $\Rightarrow$ non-singular in codimension 1".

Remark 0.3. *An 1-dimensioanal regular local ring is a DVR. (cf. [Matsumura, p.79])*

Exercise 0.4. *Show that $\dim \mathbb{A}^2_{\mathbb{C}} = 2$.*

Show that $\mathbb{A}^2_k$ is non-singular.

Let X be a variety (maybe singular), one can define $Div(X)$ to be the divisor group. Let $K(X)$ be the field of rational functions. If X is non-singular in codimension 1, (i.e. localization at height 1 prime gives a regular local ring. An equivalent condition is that the singular locus has codim ≥ 2), then one can define

$$K(X)^* \to Div(X),$$

by $div(f) = \sum_Y v(Y)$. Where the sum is taking over all codimension 1 subvarieties. The image is called *principal divisors*. Two divisor are said to be linearly equivalent if they differ by a principal divisor. We define the divisor class group $Cl(X) := Div/\{\text{principal divisors}\}$.

Exercise 0.5. *Prove that $Cl(\mathbb{P}^n) \cong \mathbb{Z}$. Hence we can define the degree of a divisor via $Div(\mathbb{P}^n) \to Cl(\mathbb{P}^n) \to \mathbb{Z}$.*

However, the divisor class group is in general not that simple. For example, $Cl(E) \ncong \mathbb{Z}$.

Every divisor on a non-singular variety is locally principal, i.e. in a sufficiently small neighborhood U_α, $D|U_\alpha = div(f_\alpha)$ for some rational function f_α. We call f_α the local equation of D. Note that on $U_\alpha \cap U_\beta$, $f_\alpha f_\beta^{-1}$ is regular. On the other hand, if one has an open covering $X = \cup U_\alpha$ and a collection of (f_α, U_α) such that $f_\alpha f_\beta^{-1}$ is regular, then this defines a divisor.

Example 0.6. *Consider $X = \mathbb{P}^1 = U_0 \cup U_1$. Let t, s be local coordinate of U_0, U_1 respectively. One has $s = t^{-1}$. Also $K(X) = k(t) = k(s)$. Now consider 1-form dt on U_0, it's clear that $dt = -ds/s^2$. We have $\{(1, U_0), (-1/s^2, U_1)\}$ which represent the 1-form. The divisor is $-2[\infty]$ which is the canonical divisor.*

Example 0.7. *Similarly, consider $X = \mathbb{P}^n = U_0 \cup ... \cup U_n$. Conputatation shows that $K_X = -(n+1)H$ for some hyperplane H.*

Remark 0.8. *It could happen that difference choices of coordinates gives different divisor. Indeed, they might give different divisor but still linear equivalent. One should say that the canonical divisor is the equivalent class of the divisor defined this way. Or sometimes we simply said that a divisor is a canonical divisor if it is in the linear equivalent class.*

For a given a divisor $D = \sum n_i D_i$, we say D is effective, denoted $D \geq 0$, if $n_i \geq 0$ for all i. The linear series of D is defined as

$$|D| := \{D' \in Div(X) | D' \sim D, D' \geq 0\}.$$

We say that D is a reduced divisor if $n_i = 1$ for all i, and D is irreducible if $n_i = 1$ for a unique i and $n_j = 0$ otherwise, that is, D is a subvariety.

One of the purpose to realize a divisor as a collection of local equation is that it enable us to define the *pull-back* of the divisor. Let $f : X \to Y$ be a morphism of two non-singular varieties. Given $D \in Div(Y)$, it associate to a collection $\{(f_\alpha, U_\alpha)\}$ for a small enough open covering $\{U_\alpha\}$. This gives an open covering $\{f^{-1}(U_\alpha)\}$ on X and also a collection of local equations $\{(f_\alpha \circ f, f^{-1}(U_\alpha))\}$. We get a divisor on X, denoted f^*D. It's clear that if $D_1 \sim D_2$ then $f^*D_1 \sim f^*D_2$ because $f^*D_1 - f^*D_2 = div(g \circ f)$ if $D_1 - D_2 = div(g)$.

Example 0.9. *Consider the morphism* $f : \mathbb{P}^2 \to \mathbb{P}^5$ *via* $f([a_0, a_1, a_2]) = [a_0^2, a_0a_1, a_0a_2, a_1^2, a_1a_2, a_2^2]$. *Let* H *be the hyperplane* $Z_0 + Z_1 = 0$*, what's the divisor* f^*H*? Let* H' *be the hyperplane* $Z_0 = 0$*, what's the divisor* f^*H'*?*