Algebraic surfaces

2. Affine varieties and projective varieties

0.1. **affine varieties.** The main object in algebraic geometry is variety. To set it up, let's first fix an algebraically closed field k. The affine n-space over k, denoted $\mathbb{A}^n_k$, is the set of all n-tuples. To study $\mathbb{A}^n$, the polynomial ring $A := k[x_1, .., x_n]$ is a convenient tool. They are closely connected via the following operation:

(1) Given a set of polynomials T, one can define $Z(T)$, the common zero locus of T. We call such $Z(T)$ an *algebraic set*.
(2) Given a subset Y of affine space, one can define $I(Y)$ which consists of polynomials vanish along Y. It's immediate that $I(Y)$ is an ideal.

These two operation give connection between ideals and algebraic sets. It's not difficult to see that one can define a topology on $\mathbb{A}^n$ with algebraic sets as closed sets. This topology is called the *Zariski topology.* Can one also construct a topology on the algebraic side? The answer is yes, with some extra care.

One notices that different ideals might give the same algebraic set, for example, the ideal (x) and (x^3) do. Among all ideals defining the same algebraic set, there is a maximal one, the *radical ideal.* This is a consequence of:

Theorem 0.1 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field and $A = k[x_1, ..., x_n]$ be the polynomial ring. Let $\mathfrak{a}$ be an ideal in A, then*

$$I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

Corollary 0.2. *There is an one-to-one correspondence between algebraic sets and radical ideals. Furthermore, the algebraic set is irreducible (resp. a point) if and only if its ideal is prime (resp. maximal).*

Definition 0.3. *An algebraic set X is irreducible it if can't be written as union of two algebraic set in a non-trivial way. More precisely, if $X = X_1 \cup X_2$ with X_i being algebraic sets, then either $X = X_1$ or $X = X_2$.*

An affine variety is an irreducible algebraic set in $\mathbb{A}^n$. An open subset of an affine variety is a quasi-affine variety.

Since A is Noetherian, it's easy to see that every algebraic set can be written as finite union of affine varieties. (This is basically what primary decomposition does). For an algebraic set X, it has the induced Zariski topology. It's easy to see that it has *descending chain condition* for closed subsets, i.e. for any sequence $Y_1 \supseteq Y_2 \supseteq \ldots$ of closed subsets, there is an integer r such that $Y_r = Y_{r+1} = ...$ A topological space is called Noetherian if ithas d.c.c for closed subsets.

With the correspondence in mind, we can define the concept of dimension geometrically and algebraically.

Definition 0.4. *For a Noetherian topological space X, the dimension of X, denoted* $\dim X$*, is defined to be the supremum (=maximum) of the length of chain of closed subvarieties.*

For an affine variety $X in \mathbb{A}^n$, the polynomial functions A restrict to X is nothing but the homomorphism $\pi : A \to A/I(X)$. The ring $A/I(X)$ is called the *coordinate ring* of X, denoted $A(X)$. One can recover the geometry of X from $A(X)$ by considering $Spec(A(X))$, which consist of prime ideals in $A(X)$. One can give the Zariski topology on $Spec(A(X))$ which is closely related to the Zariski topology on X. This is actually the construction of *affine scheme.* And affine variety can be viewed as a nice affine scheme.

Exercise 0.5. *The coordinate ring of an affine variety is a domain and a finitely generated k-algebra. Conversely, a domain which is a finitely generated k-algebra is a coordinate ring of an affine variety.*

One can also similarly define the *Krull dimension* or simply dimension to be the supremum of length of chain of prime ideals of a ring. It's easy to see that for an algebraic set X, then

$$\dim X = \dim A(X).$$

However, it's not trivial to prove that $\dim \mathbb{A}^n = n$.

0.2. **projective varieties.** A projective n-space, denoted $\mathbb{P}^n$ is defined to be the set of equivalence classes of $(n+1)$ tuples $(a_0, ..., a_n)$, with not all zero. Where the equivalent relation is $(a_0, ..., a_n) \sim (\lambda a_0, ..., \lambda a_n)$ for all $\lambda \neq 0$. We usually write the equivalence class as $[a_0, ..., a_n]$ or $(a_0 : ... : a_n)$.

One can first consider $\mathbb{P}^n$ as a quotient of $\mathbb{A}^{n+1} - \{(0, ..., 0)\}$. Let $\pi : \mathbb{A}^{n+1} - \{(0, ..., 0)\} \to \mathbb{P}^n$ be the quotient map. And we can topologize $\mathbb{P}^n$ by the quotient topology of Zariski topology. Then one sees that for a closed set $Y \subset \mathbb{P}^n$, $\pi^{-1}(Y)$ corresponds to a homogeneous ideal $I \lhd k[x_0, ..., x_n]$.

We have the similar correspondence between projective algebraic sets and homogeneous radical ideals. There is an ieal need to be excluded, the *irrelevant maximal ideal*, $(x_0, .., x_n)$.

Another important description is to give $\mathbb{P}^n$ an open covering of $n+1$ copies of $\mathbb{A}^n$. It follows that every projective variety can be covered by affine varieties.

To this end, we can simply consider

$$\imath_j : \mathbb{A}^n \to \mathbb{P}^n \text{ by } \imath_j(a_1, ..., a_n) = [a_1, ..., a_{j-1}, 1, a_j, .., a_n].$$

On the other hand, let H_j be the hyperplane $x_j = 0$ in $\mathbb{P}^n$. Then we have

$$p_j : \mathbb{P}^n - H_j \to \mathbb{A}^n \text{ by } p_j[a_0, ..., a_n] = (a_0/a_j, ..., a_{j-1}/a_j, a_{j+1}/a_j, ..., a_n/a_j).$$

Example 0.6. *We have seen that an elliptic curve E can be maps to $\mathbb{C}^2$ by the Weierstrass functions. Compose with $\imath_2$, we have a map $\varphi : E \to \mathbb{P}^2$. The defining equation in $\mathbb{C}^2$ is $y^2 = 4x^3 - g_2 x - g_3$. While the defining equation in $\mathbb{P}^2$ is the homogenized equation $y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3$*

In general, the equations in affine spaces and projective spaces are corresponding by *homogenization* and *dehomoenization*.

By a *variety*, we mean affine, quasi-affine, projective or quasi-projective variety. (More generally, an abstract variety can be defined as an *integral separated scheme of finite type over an algebraically closed field k*).