Algebraic surfaces

Ruled surfaces

Remark 0.1. *This is a combination of [Be] and [Ha].*

Definition 0.2. *A vector bundle of rank r over X is $\pi : E \to X$ such that π is locally trivial and patching together by regular functions.*

That is, there is an open covering $\mathcal{U} = \{U_i\}$ of X and trivialization maps $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}^r$ such that the transition function $\varphi_{ij} : U_i \cap U_j \to GL(r,k)$ is regular.

Note that φ_{ij} is defined as

$$\varphi_i \circ \varphi_j^{-1}(x,v) = (x, \varphi_{ij}(x)v).$$

Definition 0.3. *Let $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ be vector bundles. An morphism of vector bundles is a morphism $f : E_1 \to E_2$ such that $\pi_2 \circ f = \pi_1$.*

Exercise 0.4. *Show that*

$$H^1(X, GL(r,k)) \cong \{\textit{isomorphic classes of vector bundles of rank } r \textit{ over } X\}.$$

A rank 1 vector bundle is call a line bundle. It's clear that for a given line bundle D, one can associate a line bundle $\mathrm{L}(D)$ with transition function f_i/f_j, where f_i are the local defining equations of D. One can prove that

Exercise 0.5. *Two divisor D_1, D_2 are linearly equivalent if and only if $\mathrm{L}(D_1), \mathrm{L}(D_2)$ are isomorphic line bundles.*

Therefore, there is a injective group homomorphism

$$(Div(X)/\sim) \to H^1(X, GL(1,k)) = H^1(X, \mathcal{O}_X^*).$$

It's in fact an isomorphism . We hence call it the Picard group of X and denoted it by $\mathrm{Pic}(X)$.

Remark 0.6. *The line bundle $\mathrm{L}(D)$ induces the sheaf of sections naturally. It's easy to that the sheaf is actually $\mathcal{O}_X(D)$.*

One can similarly define a $\mathbb{P}^r$-bundle over X as we did for vector bundle. The only difference is that the transition function has value in $PGL(r+1,k) = Aut(\mathbb{P}^r)$.

Example 0.7. *A $\mathbb{P}^1$-bundle over a curve B is a ruled surface.*

Recall that by a (geometrically) ruled surface over B, we mean a surface X with a smooth morphism $\pi : X \to B$ such that each fiber is isomorphic to $\mathbb{P}^1$. Indeed, by the following theorem, one can see that a ruled surface is a $\mathbb{P}^1$-bundle.

Theorem 0.8 (Noether-Enriques). *Let X be a surface and $\pi : X \to B$ be a morphism to a curve B. If $x \in B$ is a point such that π is smooth over x and $\pi^{-1}(x) \cong \mathbb{P}^1$, then there is an Zariski open set $U \ni x$ such that $\pi^{-1}(U) \cong U \times \mathbb{P}^1$.*

One observe that there is a exact sequence of groups

$$1 \to k^* \to GL(2, k) \to PGL(2, k) \to 1.$$

Over a variety X , then we have an exact sequence of sheaves:

$$1 \to \mathcal{O}_X^* \to GL(2, k) \to PGL(2, k) \to 1.$$

As we can identify ruled surface over B with $\mathbb{P}^1$-bundle over B , then the following induced exact sequence is important and illuminating for studying ruled surface.

$$H^1(B, \mathcal{O}_B^*) \to H^1(B, GL(2, k)) \to H^1(B, PGL(2, k)) \to H^2(B, \mathcal{O}_B^*).$$

Since $\dim B = 1$, one has $H^2(B, \mathcal{O}_B^*) = 0$. One can conclude that

(1) Every $\mathbb{P}^1$-bundle over a curve B is $\mathbb{P}(E)$ for some rank 2 vector bundle E over B.
(2) $\mathbb{P}(E_1) \cong \mathbb{P}(E_2)$ if and only if $E_1 \cong E_2 \otimes L$ for some line bundle L.

It's therefore essential and convenient to study rank 2 vector bundle over a curve B.

Let $\pi : E \to B$ be a rank 2 vector bundle. By replacing E with $E \otimes L^n$ for some ample line bundle L, we may assume that $h^0(B, E) > 0$. Thus one has $0 \to \mathcal{O}_B \to E$. (We abuse the notation of sheaves of line bundles(divisors) and line bundles). Let Q be the quotient. It's clear that Q is a sheaf generically of rank 1. Take $Q^{\vee\vee}$, it's then a line bundle. *Projectivize* $E \to Q^{\vee\vee} \to 0$, one has

$$\sigma : X \cong \mathbb{P}(Q) \to \mathbb{P}(E).$$

This is called a section of the ruled surface $\pi : \mathbb{P}(E) \to X$. It's clear that $\pi \circ \sigma = id_X$. However, this depends on choice of n.

Another way around this is via the *tautological bundle*. Let $\pi : \mathbb{P}(E) \to B$ be a ruled surface. One has naturally on $\mathbb{P}(E)$

$$0 \to N \to \pi^* E \to Q \to 0.$$

Q is called the tautological line bundle and denoted by $\mathcal{O}_{\mathbb{P}(E)}(1)$. Q is associated to a divisor C such that $C.F = 1$.

Proposition 0.9. *Let $\pi : X = \mathbb{P}(E) \to B$ be the $\mathbb{P}^1$-bundle as before. We keep the notation as above. We have*

(1) $\mathrm{Pic} X = \pi^* \mathrm{Pic} B \oplus \mathbb{Z}[C]$.
(2) $\mathrm{Num}(X) = \mathbb{Z}[C] \oplus \mathbb{Z}[F]$.
(3) $C^2 = deg(E)$. *Recall that* $deg(E) := deg(\wedge^2 E)$.
(4) $K_X \sim -2C + (deg(E)F + \pi^* K_B)$.

Proof. (1) For a given divisor $D \in Div(X)$, let $m = D.F$. We consider $D' = D - mC$, it's clear that $D'.F = 0$. We claim that D' is linearly equivalent to π^*G for some $G \in Div(B)$.

To this end, we consider $D_n := D' + nF$ and let n go to infinity. Note that $D_n^2 = D^2$, $F.K_X = -2$, and hence $D_n.K_X = D.K_X - 2n$. By Riemann-Roch,

$$\chi(X, \mathcal{O}(D_n)) = \chi(X, \mathcal{O}_X) + \frac{1}{2}(D^2 - D.K_X + 2n) > 0, \forall n \gg 0.$$

On the other hand, $h^2(X, \mathcal{O}(D_n) = h^0(X, \mathcal{O}(K_X - D_n)) = 0$ for $n \gg 0$. Thus, D_n is effective for $n \gg 0$. Pick any $\Delta \in |D_n|$, one sees that $\Delta.F = 0$ and hence $\pi(\Delta) \neq B$. If follows that $\Delta = \pi^*G'$ for some $G' \in Div(B)$. And therefore $D \sim \Delta - nF = \pi^*(G' - n\pi(F))$.

(2) By (1), one has $\mathrm{Num}X = \mathbb{Z}[C] \oplus \pi^*\mathrm{Num}B$. Since $deg : \mathrm{Num}B \to \mathbb{Z}$ is an isomorphism. We are done.

(3) This is more subtle which we will treat later. We need some thing about the Chern classes in order to prove this. For basic notions, please see the appendix following the proof.

We consider the exact sequence

$$0 \to N \to \pi^*E \to Q \to 0,$$

on X.

One has

$$\pi^*c_1(\wedge^2 E) = \pi^*c_1(E) = c_1(\pi^*E) = c_1(N) + c_1(Q).$$

And

$$c_1(N).c_1(Q) = c_2(\pi^*E) = \pi^*c_2(E) = 0$$

because $c_2(E) \in H^4(B, \mathbb{Z}) = 0$. It follows that

$$C^2 = c_1(Q).c_1(Q) = c_1(Q).c_1(\pi^* \wedge^2 E) - c_1(Q).c_1(N).$$

Since $Q = L(C)$ and $\wedge^2 E = L(\Delta)$ for some divisor Δ of degree $\deg(E)$. We have $C^2 = C.\pi^*\Delta = deg(E)$.

(4) Since $K_X.F = -2$, by (1) one has that $K_X \sim -2C + \pi^*E$ for some E. In NumX, $[K_X] = -2[C] + b[F]$. And let Γ be a section of $\pi : X \to B$, then $[\Gamma] = [C] + r[F]$. By adjunction formula,

$$2g(B) - 2 = 2g(\Gamma) - 2 = -deg(E) + b.$$

Fix a point $P_0 \in B$, and fix F_0 to be the fiber over P_0 then $(K_X + 2C - \pi^*K_B - deg(E)\pi^*B_0).F = 0$. By (1),

$$K_X \sim -2C + \pi^*K_B + deg(E)\pi^*B_0 + \pi^*\Delta,$$

for some Δ with degree $= 0$.

$\square$

Let X be a non-singular projective variety of dimension n and let E be a vector bundle of rank r on X. One can define *Chern classes* $c_i(E) \in H^{2i}(X, \mathbb{Z})$ for $i = 0, .., r$. Moreover, one can have *total Chern class*

$$c_0(E) + c_1(E) + c_2(E) + ... + c_r(E)$$

and *Chern polynomial*

$$c_t(E) := c_0(E) + c_1(E)t + c_2(E)t^2 + ... + c_r(E)t^r$$

satisfying:

(1) If $E = L(D)$ is a line bundle induced from a divisor D, then $c_t(E) = 1 + [D]t$, where $[D]$ denote the image of D in $Div(X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$.

(2) If $f : Y \to X$ is a morphism of non-singular varieties, then $c_i(\pi^* E) = \pi^*(c_i(E))$.

(3) If $0 \to E_1 \to E \to E_2 \to 0$ is an exact sequence of vector bundles, then

$$c_t(E) = c_t(E_1)c_t(E_2),$$

where the product of classes are the cup product.