Algebraic surfaces

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1. Comapct Riemann surface and complex algebraic curves

This section is devoted to illustrate the connection between compact Riemann surface and complex algebraic curve. We will basically work out the example of elliptic curves and leave the general discussion for interested reader.

Let $\Lambda \subset \mathbb{C}$ be a lattice, that is, $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ with $\mathbb{C} = \mathbb{R}\lambda_1 + \mathbb{R}\lambda_2$. Then an elliptic curve $E := \mathbb{C}/\Lambda$ is an abelian group.

On the other hand, it's a compact Riemann surface. One would like to ask whether there are holomorphic function or meromorphic functions on E or not.

To this end, let $\pi : \mathbb{C} \to E$ be the natural map, which is a group homomorphism (algebra), holomorphic function (complex analysis), and covering (topology). A function $\bar{f}$ on E gives a function f on $\mathbb{C}$ which is *double periodic*, i.e.,

$$f(z+\lambda_1) = f(z), f(z+\lambda_2) = f(z), \forall z \in \mathbb{C}.$$

Recall that we have the following

Theorem 1.1 (Liouville)**.** *There is no non-constant bounded entire function.*

Corollary 1.2. *There is no non-constant holomorphic function on E.*

Proof. First note that if $\bar{f}$ is a holomorphic function on E, then it induces a double periodic holomorphic function on $\mathbb{C}$. It then suffices to claim that a a double periodic holomorphic function on $\mathbb{C}$ is bounded.

To do this, consider $\mathcal{D} := \{z = \alpha_1\lambda_1 + \alpha_2\lambda_2 | 0 \leq \alpha_i \leq 1\}$. The image of f is $f(D)$. However, D is compact, so is $f(D)$ and hence $f(D)$ is bounded. By Liouville theorem, f is constant and hence so is $\bar{f}$. □

The next hope is to ask if there is a meromorphic function on E with only a simple pole or not. The answer is NO, which can be proved by residue theorem.

Exercise 1.3. *Prove that there is no non-zero meromorphic function on E with only a simple pole*

Then the next step is to look for functions with pole of order 2. Luckily, we have one, which is the Weierstrass $\mathcal{P}$-function,

$$\mathcal{P}(z) := z^{-2} + \sum_{\omega \in \Lambda - \{0\}} ((z-\omega)^{-2} - \omega^{-2}).$$

And $\mathcal{P}'$ is a functions with pole of order 3. By direct computation, one sees that

Lemma 1.4.

$$\mathcal{P}'(z)^2 = 4\mathcal{P}(z)^3 - g_2\mathcal{P}(z) - g_3.$$

Where

$$g_2 = 60 \sum_{\omega \in \Lambda - \{0\}} \omega^{-4}$$

and

$$g_3 = 140 \sum_{\omega \in \Lambda - \{0\}} \omega^{-6}.$$

By considering the map $E \to \mathbb{C}^2$ (or to $\mathbb{P}^2$) given by $\bar{z} \mapsto (\mathcal{P}(z), \mathcal{P}'(z))$, then one realize the elliptic curve as a cubic curve in $\mathbb{P}^2$. The affine defining equation in $\mathbb{C}^2$ is

$$y^2 = 4x^3 - g_2x - g_3.$$

And the projective defining equation is

$$y^2z = 4x^3 - g_2xz^2 - g_3z^3.$$

We therefore realize an elliptic curve as an algebraic curve.

Similar phenomena occurs for any compact Riemann surface. As the above example suggested, the essential point is find enough functions and then determine the algebraic relation between those functions. All these can be done for any compact Riemann surface. Thus we have the following

Theorem 1.5. *Any compact Riemann surface can be embedded into a projective space as an algebraic curve.*

To study functions more systematically, it's natural to consider *divisors*.

Definition 1.6. *Let X be a compact Riemann surface/ non-singular algebraic curve, a divisor, denoted $D = \sum n_i P_i$, is a finite formal sum of points (codim=1).*

The idea for divisors is to collect information on poles and zeros. We denote the functions with prescribed poles and zero as

$$\mathcal{L}(D) := \{f \in \mathcal{M}(X) | div(f) + D \geq 0\}.$$

It's clearly a vector space over the ground field and its dimension is denoted $l(D)$. Another important notion for divisor is the *degree*, which is

$$deg(D) := \sum n_i.$$

Let's look at the definition a little bit more. Given a meromorphic function (or rational function) $f(z)$, one can collect all the poles and zeros together with orders at these points, give rise to a divisor $div(f)$.

Example 1.7. *Let $X = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Let $f_1(z) = z, f_2(z) = 1/(z-1), f_3(z) = z/(z-1)^2$. By easy computation, one finds that $div(f_1) = 1[0]-1[\infty]$ since it has a zero at 0 and a pole at ∞. Similarly, $div(f_2) = 1[\infty]-1[1]$, $div(f_3) = 1[0] + 1[\infty] - 2[1]$.*

Now fix a divisor $D = 2[1]$. What is $\mathcal{L}(D)$? What does it mean? It is nothing but the set of meromorphic functions with at most a pole of order 2 at $[1]$ and no other poles. More precisely,

$$\mathcal{L}(D) = \{g(z)/(z-1)^2 | deg(g(z)) \leq 2\}.$$

Because if $deg(g(z)) \geq 3$ then it gives a pole at ∞.

Now we can state the most important theorem for curves, the Riemann-Roch theorem:

Theorem 1.8 (Riemann-Roch)**.**

$$l(D) - l(K-D) = deg(D) + 1 - g(X),$$

where K denotes the canonical divisor, and $g(X)$ is the genus of the curve X.

We will turn to the discussion of canonical divisor more thoroughly later. At this moment, it's enough to know that it's a divisor of degree $2g-2$ and for an elliptic curve, $K = 0$.

Before we move on, we recall an easy fact:

Proposition 1.9. *1 Let $D = \sum n_i P_i$ be a divisor. We said that D is effective if $n_i \geq 0$ for all i, denoted $D \geq 0$. Suppose now that D is an effective non-zero divisor then $\mathcal{L}(D) = \{0\}$.*

Proof. Exercise. (Hint: prove that a holomorphic function on a compact Riemann surface must be constant) □

We now redo the example of elliptic curve with the help of Riemann-Roch theorem. It's a fact that the following argument also works for any compact Riemann surface.

Example 1.10. *Let $E = \mathbb{C}/\Lambda$ be an elliptic curve. Then the genus is* 1. *Let* $\bar{0}$ *be the image of* $\pi(0)$ *in* E. *By Prop. 1.91 and* $K = 0$, *Riemann-Roch reads:*

$$l(D) = deg(D).$$

The vector space $\mathcal{L}(k\bar{0})$ *has a natural basis* $\{1, \mathcal{P}\}$, $\{1, \mathcal{P}, \mathcal{P}'\}$ *respectively when* $k = 2, 3$. *However, when* $k = 6$, *one finds that* $\{1, \mathcal{P}, \mathcal{P}', \mathcal{P}^2, \mathcal{P}\mathcal{P}', \mathcal{P}^3, \mathcal{P}'^2\}$ *must be linearly depedent. Hence there must be a relation between them involving* $\mathcal{P}^3, \mathcal{P}'^2$.

On the other hand, for a complex algebraic curve X, it is non-singular if and only if one can find a local chart at each point. Hence a non-singular complex algebraic curve is a Riemann surface. If X is projective, then it's a closed subset in a compact set $\mathbb{P}^N$. Therefore it is compact.

We've seen that an elliptic curve E can be embedded into $\mathbb{P}^2$ as a non-singular cubic curve. There is a group structure on E. How this group structure behave on cubic curves? In fact there is a nice geometric correspondence via intersection.