Algebraic Surfaces

Homework 2

Let X be a non-singular surface. Recall that a divisor D on$X$ is ample (resp. nef) if and only if $D.C > 0$ (resp. ≥ 0) and $D^2 > 0$ (resp. ≥ 0).

(1) Let C be an irreducible curve and $D = \sum n_i D_i$ be an effective divisor. Prove that if $C.D < 0$ then $C = C_i$ for some i and $C^2 < 0$.

(2) Let C be a singular curve in X. By blowing-up of X along singularities of C. One has $\pi : \widetilde{X} \to X$ such that the proper transform $\widetilde{C}$ is non-singular. Prove that $p_a(\widetilde{C}) < p_a(C)$.
(hint: $x \in C$ is singular if and only if $m_x(C) \geq 2$, where $m_x(C)$ is the multiplicity of C at x.

(3) Let L be an ample divisor on X. And let $\pi : \widetilde{X} = Bl(X) \to X$ be a blowing-up. Show that $\pi^* L$ is not ample but it's nef and big.

(4) Let L be an ample divisor on X. Show that $|K_X + mL|$ is base point free for $m \gg 0$.
(hint: the inequality for any two effective divisor

$$C.D \geq \sum_{x \in C \cap D} m_x(C) m_x(D)$$

might be useful.)