3 Bivariate Transformations

Let (X, Y) be a bivariate random vector with a known probability distribution. Let $U = g_1(X, Y)$ and $V = g_2(X, Y)$, where $g_1(x, y)$ and $g_2(x, y)$ are some specified functions. If B is any subset of $\mathbb{R}^2$, then $(U, V) \in B$ if and only if $(X, Y) \in A$, where $A = \{(x, y) : (g_1(x, y), g_2(x, y)) \in B\}$. Thus $P((U, V) \in B) = P((X, Y) \in A)$, and the probability of (U, V) is completely determined by the probability distribution of (X, Y).

If (X, Y) is a discrete bivariate random vector, then

$$f_{U,V}(u, v) = P(U = u, V = v) = P((X, Y) \in A_{u,v}) = \sum_{(x,y)\in A_{uv}} f_{X,Y}(x, y),$$

where $A_{u,v} = \{(x, y) : g_1(x, y) = u, g_2(x, y) = v\}$.

Example 3.1 *(Distribution of the sum of Poisson variables) Let X and Y be independent Poisson random variables with parameters θ and λ, respectively. Thus, the joint pmf of (X, Y) is*

$$f_{X,Y}(x, y) = \frac{\theta^x e^{-\theta}}{x!} \frac{\lambda^y e^{-\lambda}}{y!}, \quad x = 0, 1, 2, \ldots, \quad y = 0, 1, 2, \ldots$$

Now define $U = X + Y$ and $V = Y$, thus,

$$f_{U,V}(u, v) = f_{X,V}(u - v, v) = \frac{\theta^{u-v} e^{-\theta}}{(u - v)!} \frac{\lambda^v e^{-\lambda}}{v!}, \quad v = 0, 1, 2, \ldots, \quad u = v, v + 1, \ldots$$

The marginal of U is

$$\begin{aligned} f_U(u) &= \sum_{v=0}^{u} \frac{\theta^{u-v} e^{-\theta}}{(u - v)!} \frac{\lambda^v e^{-\lambda}}{v!} = e^{-(\theta+\lambda)} \sum_{v=0}^{u} \frac{\theta^{u-v}}{(u - v)!} \frac{\lambda^v}{v!} \\ &= \frac{e^{-(\theta+\lambda)}}{u!} \sum_{v=0}^{u} \binom{u}{v} \lambda^v \theta^{u-v} = \frac{e^{-(\theta+\lambda)}}{u!} (\theta + \lambda)^u, \quad u = 0, 1, 2, \ldots \end{aligned}$$

This is the pmf of a Poisson random variable with parameter $\theta + \lambda$.

Theorem 3.1 *If $X \sim$ Poisson(θ) and $Y \sim$ Poisson(λ) and X and Y are independent, then $X + Y \sim$ Poisson$(\theta + \lambda)$.*

If (X, Y) is a continuous random vector with joint pdf $f_{X,Y}(x, y)$, then the joint pdf of (U, V) can be expressed in terms of $F_{X,Y}(x, y)$ in a similar way. As before, let $A = \{(x, y) : f_{X,Y}(x, y) > 0\}$ and $B = \{(u, v) : u = g_1(x, y)$ and $v = g_2(x, y)$ for some $(x, y) \in A\}$. For the simplest version of this result, we assume the transformation $u = g_1(x, y)$ and $v = g_2(x, y)$ defines a one-to-one transformation of A to B. For such a one-to-one, onto transformation, we can solve the equations

$u = g_1(x, y)$ and $v = g_2(x, y)$ for x and y in terms of u and v. We will denote this inverse transformation by $x = h_1(u, v)$ and $y = h_2(u, v)$. The role played by a derivative in the univariate case is now played by a quantity called the Jacobian of the transformation. It is defined by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix},$$

where $\frac{\partial x}{\partial u} = \frac{\partial h_1(u,v)}{\partial u}$, $\frac{\partial x}{\partial v} = \frac{\partial h_1(u,v)}{\partial v}$, $\frac{\partial y}{\partial u} = \frac{\partial h_2(u,v)}{\partial u}$, and $\frac{\partial y}{\partial v} = \frac{\partial h_2(u,v)}{\partial v}$.

We assume that J is not identically 0 on B. Then the joint pdf of (U, V) is 0 outside the set B and on the set B is given by

$$f_{U,V}(u, v) = f_{X,Y}(h_1(u, v), h_2(u, v))|J|,$$

where $|J|$ is the absolute value of J.

Example 3.2 *(Sum and difference of normal variables) Let X and Y be independent, standard normal variables. Consider the transformation $U = X + Y$ and $V = X - Y$. The joint pdf of X and Y is, of course,*

$$f_{X,Y}(x, y) = (2\pi)^{-1} \exp(-x^2/2) \exp(-y^2/2), \quad -\infty < x < \infty, -\infty < y < \infty.$$

so the set $A = \mathbb{R}^2$. Solving the following equations

$$u = x + y \quad \textit{and} \quad v = x - y$$

for x and y, we have

$$x = h_1(x, y) = \frac{u + v}{2}, \quad \textit{and} \quad y = h_2(x, y) = \frac{u - v}{2}.$$

Since the solution is unique, we can see that the transformation is one-to-one, onto transformation from A to $B = \mathbb{R}^2$.

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

So the joint pdf of (U, V) is

$$f_{U,V}(u, v) = f_{X,Y}(h_1(u, v), h_2(u, v))|J| = \frac{1}{2\pi} e^{-((u+v)/2)^2/2} e^{-((u-v)/2)^2/2} \frac{1}{2}$$

for $-\infty < u < \infty$ and $-\infty < v < \infty$. After some simplification and rearrangement we obtain

$$f_{U,V}(u, v) = (\frac{1}{\sqrt{2p}\sqrt{2}} e^{-u^2/4})(\frac{1}{\sqrt{2p}\sqrt{2}} e^{-v^2/4}).$$

The joint pdf has factored into a function of u and a function of v. That implies U and V are independent.

Theorem 3.2 *Let X and Y be independent random variables. Let $g(x)$ be a function only of x and $h(y)$ be a function only of y. Then the random variables $U = g(X)$ and $V = h(Y)$ are independent.*

PROOF: We will prove the theorem assuming U and V are continuous random variables. For any $u \in mR$ and $v \in \mathbb{R}$, define

$$A_u = \{x : g(x) \leq u\} \quad \text{and} \quad B_u = \{y : h(y) \leq v\}.$$

Then the joint cdf of (U, V) is

$$\begin{aligned} F_{U,V}(u, v) &= P(U \leq u, V \leq v) \\ &= P(X \in A_u, Y \in B_v) \\ & P(X \in A_u)P(Y \in B_v). \end{aligned}$$

The joint pdf of (U, V) is

$$f_{U,V}(u, v) = \frac{\partial^2}{\partial u \partial v} F_{U,V}(u, v) = (\frac{d}{du} P(X \in A_u))(\frac{d}{dv} P(Y \in B_v)),$$

where the first factor is a function only of u and the second factor is a function only of v. Hence, U and V are independent. □

In many situations, the transformation of interest is not one-to-one. Just as Theorem 2.1.8 (textbook) generalized the univariate method to many-to-one functions, the same can be done here. As before, $\mathcal{A} = \{(x, y) : f_{X,Y}(x, y) > 0\}$. Suppose $A_0, A_1, \ldots, A_k$ form a partition of $\mathcal{A}$ with these properties. The set A_0, which may be empty, satisfies $P((X, Y) \in A_0) = 0$. The transformation $U = g_1(X, Y)$ and $V = g_2(X, Y)$ is a one-to-one transformation from A_i onto B for each $i = 1, 2, \ldots, k$. Then for each i, the inverse function from B to A_i can be found. Denote the ith inverse by $x = h_{1i}(u, v)$ and $y = h_{2i}(u, v)$. Let J_i denote the Jacobian computed from the ith inverse. Then assuming that these Jacobians do not vanish identically on B, we have

$$f_{U,V}(u, v) = \sum_{i=1}^{k} f_{X,Y}(h_{1i}(u, v), h_{2i}(u, v))|J_i|.$$

Example 3.3 *(Distribution of the ratio of normal variables) Let X and Y be independent $N(0, 1)$ random variable. Consider the transformation $U = X/Y$ and $V = |Y|$. (U and V can be defined to be any value, say (1,1), if $Y = 0$ since $P(Y = 0) = 0$.) This transformation is not one-to-one, since the points (x, y) and $(-x, -y)$ are both mapped into the same (u, v) point. Let*

$$A_1 = \{(x, y) : y > 0\}, \quad A_2 = \{(x, y) : y < 0\}, \quad A_0 = \{(x, y) : y = 0\}.$$

A_0, A_1 and A_2 form a partition of $\mathcal{A} = \mathbb{R}^2$ and $P(A_0) = 0$. The inverse transformations from B to A_1 and B to A_2 are given by

$$x = h_{11}(u,v) = uv, \quad y = h_{21}(u,v) = v,$$

and

$$x = h_{12}(u,v) = -uv, \quad y = h_{22}(u,v) = -v.$$

The Jacobians from the two inverses are $J_1 = J_2 = v$. Using

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2},$$

we have

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{2\pi} e^{-(uv)^2/2} e^{-v^2/2} |v| + \frac{1}{2\pi} e^{-(-uv)^2/2} e^{-(-v)^2/2} |v| \\ &= \frac{v}{\pi} e^{-(u^2+1)v^2/2}, \quad -\infty < u < \infty, \quad 0 < v < \infty. \end{aligned}$$

From this the marginal pdf of U can be computed to be

$$\begin{aligned} f_U(u) &= \int_0^\infty \frac{v}{\pi} e^{-(u^2+1)v^2/2} dv \\ &= \frac{1}{2\pi} \int_0^\infty e^{-(u^2+1)z/2} dz \qquad (z = v^2) \\ &= \frac{1}{\pi(u^2+1)} \end{aligned}$$

So we see that the ratio of two independent standard normal random variable is a Cauchy random variable.